

MATH 105 HW 10

Rudin 9.12

Let $0 < a < b$ and $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$

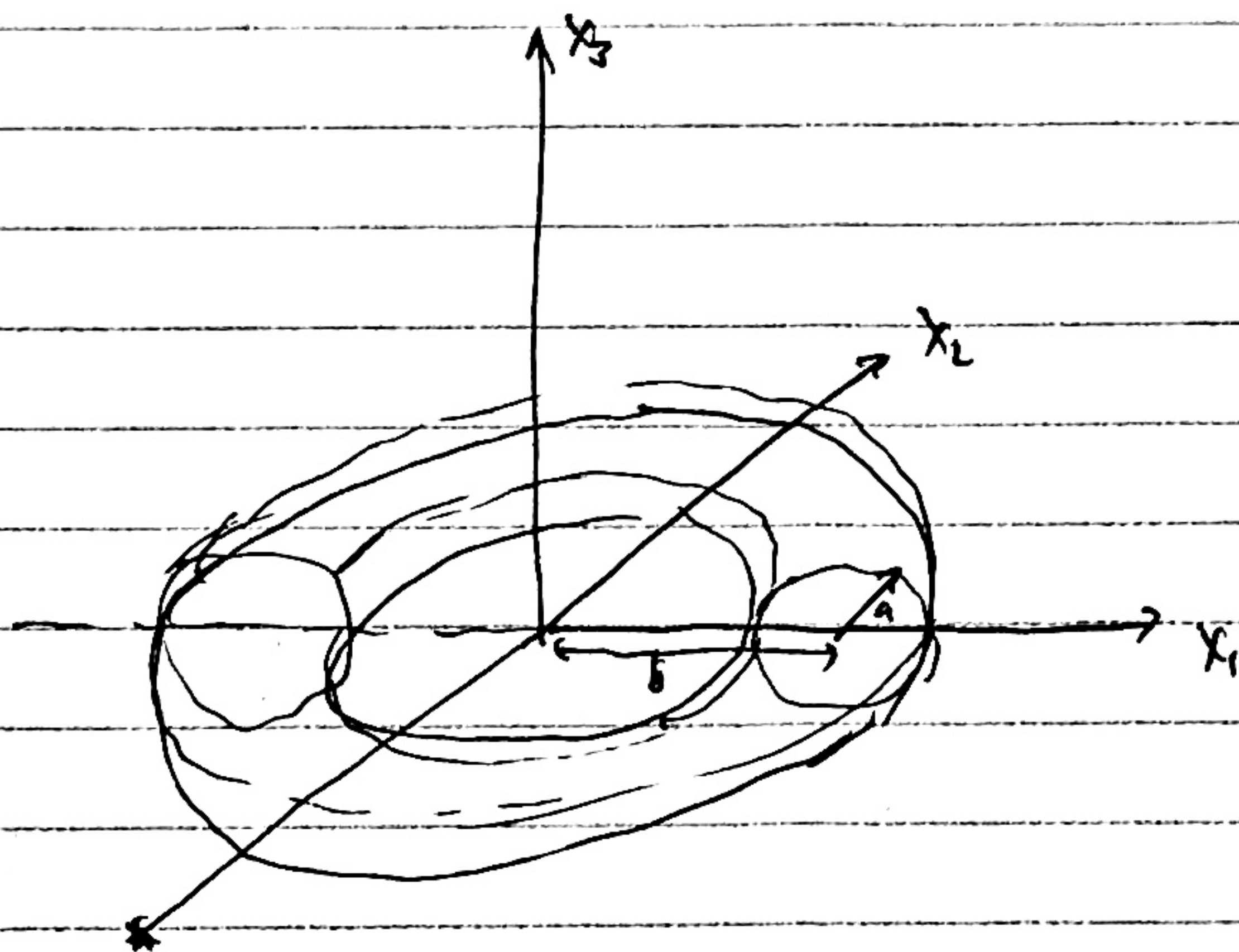
$$f_1(s, t) = (b + a \cos s) \cos t$$

$$f_2(s, t) = (b + a \cos s) \sin t$$

$$f_3(s, t) = a \sin s$$

From the expressions, $f_1^2 + f_2^2 = (b + a \cos s)^2 \Rightarrow (\sqrt{f_1^2 + f_2^2} - b)^2 = (a \cos s)^2$

$$\Rightarrow (\sqrt{f_1^2 + f_2^2} - b)^2 + f_3^2 = a^2$$



It describes a donut in 3D with diameter $2b$ (i.e. inner ring $b-a$ and outer ring $b+a$) and width diameter $2a$.

$$a) (\nabla f_1) = \begin{bmatrix} \frac{\partial f_1}{\partial s} \\ \frac{\partial f_1}{\partial t} \end{bmatrix} = \begin{bmatrix} -a \sin s \cos t \\ -(b + a \cos s) \sin t \end{bmatrix}$$

$$\text{For } \nabla f_1 = 0, \quad -a \sin s \cos t = 0 \\ \text{and } -(b + a \cos s) \sin t = 0.$$

If $\cos t = 0$, then $\sin t \neq 0 \Rightarrow b + a \cos s = 0 \Rightarrow \cos s = -\frac{b}{a}$ (contradiction since $b > a$)
 $\Rightarrow |\cos s| > 1$
 $\therefore \sin s = 0 \Rightarrow \sin t = 0.$

$$\text{Hence, } f^{-1}(p) = \begin{bmatrix} s \\ t \end{bmatrix} \text{ s.t. } \sin s = 0, \sin t = 0. \Rightarrow f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} (b + a \cos s) \cos t \\ 0 \\ 0 \end{bmatrix}$$

Since $\sin s = 0$, $\cos s = \pm 1$. Similarly, $\cos t = \pm 1$

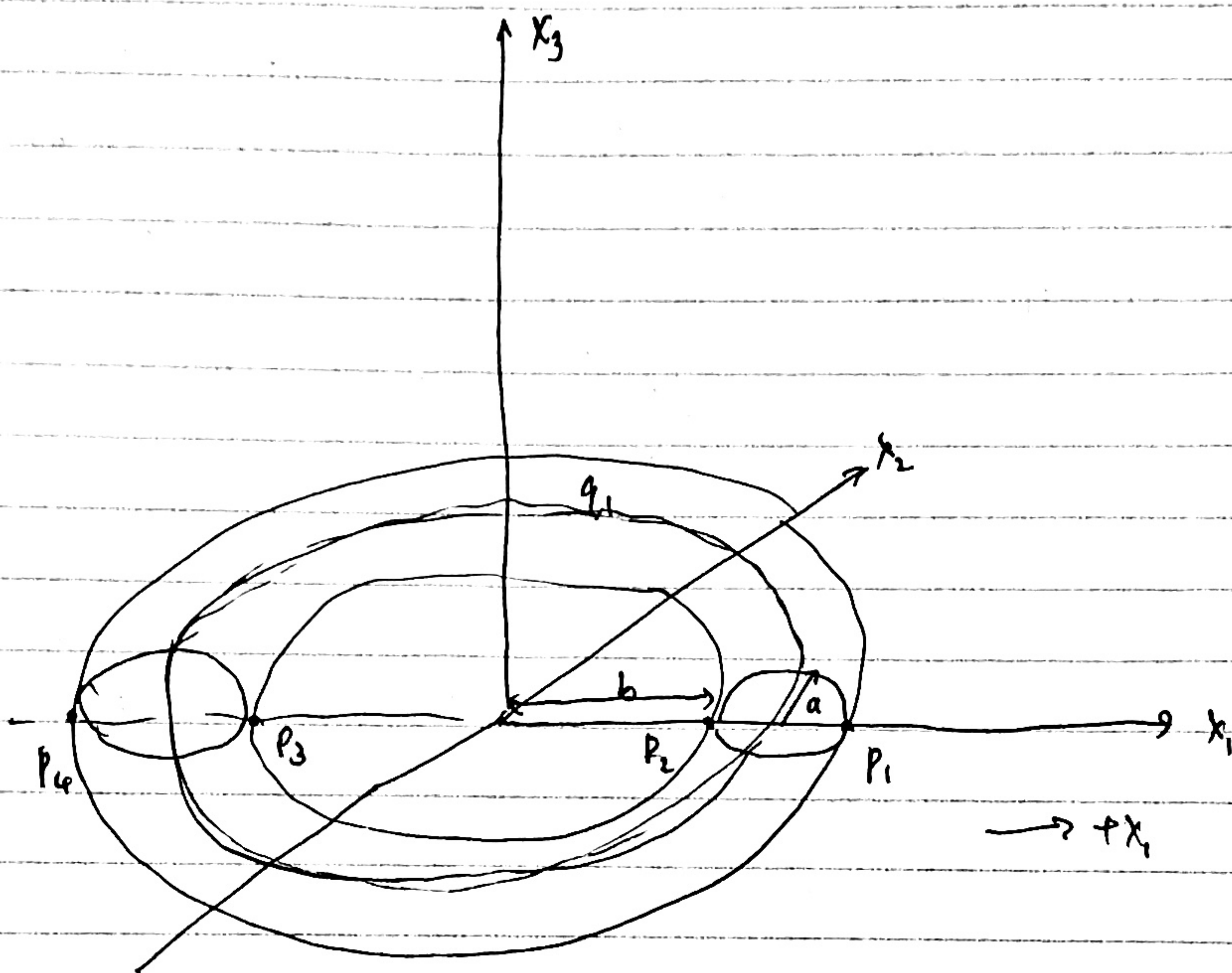
$$\Rightarrow f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) \text{ takes on 4 possible values } \begin{bmatrix} b-a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -a-b \\ 0 \\ 0 \end{bmatrix}.$$

$$\therefore p \in \left\{ \begin{bmatrix} b-a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -a-b \\ 0 \\ 0 \end{bmatrix} \right\}.$$

$$b) (\nabla f_3) = \begin{bmatrix} \frac{\partial f_3}{\partial s} \\ \frac{\partial f_3}{\partial t} \end{bmatrix} = \begin{bmatrix} a \cos s \\ 0 \end{bmatrix} \Rightarrow a \cos s = 0 \Rightarrow s = \frac{1}{2}\pi + k\pi, k \in \mathbb{Z}$$

$$\therefore f^{-1}(q) = \begin{bmatrix} \frac{1}{2}\pi + k\pi \\ t \end{bmatrix} \text{ for some } k \in \mathbb{Z}, t \in \mathbb{R} \Rightarrow q \in \left\{ \begin{bmatrix} b \cos t \\ b \sin t \\ a \end{bmatrix}, \begin{bmatrix} b \cos t \\ b \sin t \\ -a \end{bmatrix} \right\} \\ t \in \mathbb{R}$$

1 (1)



Consider $\nabla^2 f_1 = \begin{bmatrix} \frac{\partial^2 f_1}{\partial s^2} & \frac{\partial^2 f_1}{\partial s \partial e} \\ \frac{\partial^2 f_1}{\partial e \partial s} & \frac{\partial^2 f_1}{\partial e^2} \end{bmatrix} = \begin{bmatrix} -a \cos s \cos t & a \sin s \sin t \\ a \sin s \sin t & -(b + a \cos s) \cos t \end{bmatrix}$

For maximum point, $\nabla^2 f_1$ must be negative semi-definite, i.e.

$-a \cos s \cos t \leq 0$

$\det \nabla^2 f_1 = (-a \cos s \cos t)(-(b + a \cos s) \cos t) - (a \sin s \sin t)^2 \geq 0$

by Sylvester's rule of inertia

$\Rightarrow (\cos s, \cos t) = (1, 1)$ (only combination that satisfies this) $\Rightarrow \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}$ is maximum

Similarly, for minimum point, $\nabla^2 f_1$ must be positive semi-definite i.e.

$-a \cos s \cos t \geq 0$

by Sylvester's rule of inertia.

$\det \nabla^2 f_1 \geq 0$

Only $(\cos s, \cos t) = (1, -1)$ satisfies this $\Rightarrow \begin{bmatrix} -a-b \\ 0 \\ 0 \end{bmatrix}$ is minimum.

Geometrically, it is also very easy to tell that these two points are indeed the maximum and minimum of the x_1 components.

$\nabla^2 f_3 = \begin{bmatrix} -a \sin s & 0 \\ 0 & 0 \end{bmatrix}$

We already know $\sin s = \pm 1$ (since $\cos s = 0$)

hence, if $\sin s = 1$, $\nabla^2 f_3$ is negative definite \Rightarrow maximum point

if $\sin s = -1$, $\nabla^2 f_3$ is positive definite \Rightarrow minimum point.

(K)
continuedHence all points $\left\{ \begin{bmatrix} b \cos t \\ b \sin t \\ a \end{bmatrix} : t \in \mathbb{R} \right\}$ are maximum pointsand all points $\left\{ \begin{bmatrix} b \cos t \\ b \sin t \\ -a \end{bmatrix} : t \in \mathbb{R} \right\}$ are minimum points

Geometrically, the first set is the filigree of donut (upper lining) while the second set is the lower lining.

Rudin 9.13

$$f: \mathbb{R} \rightarrow \mathbb{R}^3 \text{ let } f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

$$\text{Then } f'(t) = \begin{bmatrix} \frac{d}{dt} f_1(t) \\ \frac{d}{dt} f_2(t) \\ \frac{d}{dt} f_3(t) \end{bmatrix}$$

$$|f(t)| = 1 \Rightarrow \sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2} = 1$$

$$\Rightarrow \frac{d}{dt} \left(\sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2} \right) = 0$$

$$\Rightarrow \frac{1}{2\sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2}} (2f_1(t)f_1'(t) + 2f_2(t)f_2'(t) + 2f_3(t)f_3'(t)) = 0$$

$$\Rightarrow f_1(t)f_1'(t) + f_2(t)f_2'(t) + f_3(t)f_3'(t) = 0$$

$$\therefore f'(t) \cdot f(t) = \frac{d}{dt} f_1(t) \cdot f_1(t) + \frac{d}{dt} f_2(t) \cdot f_2(t) + \frac{d}{dt} f_3(t) \cdot f_3(t)$$

$$= f_1'(t)f_1(t) + f_2'(t)f_2(t) + f_3'(t)f_3(t) = 0 \text{ as desired.}$$

Rudin 9.14

$$f(0,0) = 0; f(x,y) = \frac{x^3}{x^2+y^2} \text{ for } (x,y) \neq (0,0)$$

$$(a) \text{ AT } (x,y) \neq (0,0), D_1 f = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(\frac{x^3}{x^2+y^2} \right) = \frac{(x^2+y^2)3x^2 - x^3 \cdot 2x}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}$$

$$\text{AT } (x,y) = (0,0), \left. \frac{\partial f}{\partial x} \right|_{(0,0)} = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{f(h,0)}{h} = \lim_{h \rightarrow 0} \frac{h^3}{h^2} = 1$$

$$\text{Note } x^4 + 3x^2y^2 < 2x^4 + 4x^2y^2 + 2y^4 = 2(x^2+y^2)^2$$

$$\Rightarrow \left| \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2} \right| < \frac{2x^4 + 4x^2y^2 + 2y^4}{(x^2+y^2)^2} = 2 \text{ hence } \boxed{D_1 f \text{ is bounded.}}$$

$$(b) \text{ AT } (x,y) \neq (0,0), D_2 f = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(\frac{x^3}{x^2+y^2} \right) = \frac{(x^2+y^2) - x^3 \cdot 2y}{(x^2+y^2)^2} = -\frac{2x^3y}{(x^2+y^2)^2}$$

$$\text{AT } (x,y) = (0,0), \left. \frac{\partial f}{\partial y} \right|_{(0,0)} = \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0 \quad \left. \vphantom{\lim_{h \rightarrow 0} \frac{0}{h}} \right\} \text{ next page.}$$

Note $\left| \frac{2x^3y}{(x^2+y^2)^2} \right| < 1$. This is because $x^4 + 2x^2y^2 \geq 2\sqrt{x^4x^2y^2} \stackrel{\text{AM-GM}}{=} 2|x^3y|$

Hence $\left| \frac{2x^3y}{(x^2+y^2)^2} \right| < 1 \Rightarrow \boxed{D_2 f \text{ is bounded.}}$

f is therefore continuous because for $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M_1 + M_2}$ where M_1 and M_2 are the bounds above.

Then $d((x_1, y_1), (x_2, y_2)) < \delta$

$$\Leftrightarrow |f(x_1, y_1) - f(x_2, y_2)| = |f(x_1, y_1) - f(x_2, y_1) + f(x_2, y_1) - f(x_2, y_2)|$$

$$\leq |f(x_1, y_1) - f(x_2, y_1)| + |f(x_2, y_1) - f(x_2, y_2)|$$

Mean Value

$$< \delta \cdot M_1 + \delta \cdot M_2 = \delta (M_1 + M_2) = \frac{\varepsilon}{M_1 + M_2} (M_1 + M_2) = \varepsilon \Rightarrow f \text{ is continuous as desired.}$$

3 (b) Let $u = \begin{bmatrix} a \\ b \end{bmatrix}$. Then $D_u f(0,0) = \lim_{h \rightarrow 0} \frac{f(ha, hb) - f(0,0)}{h}$
 with $a^2 + b^2 = 1$

$$= \lim_{h \rightarrow 0} \frac{h^3 a^3}{h^2 a^2 + h^2 b^2} = a^3 \quad (\text{since } a^2 + b^2 = 1)$$

$$\therefore D_u f(0,0) \text{ exists.}$$

Since $a^2 + b^2 = 1$, $a^2 \leq 1 \Rightarrow |a| \leq 1 \Rightarrow |a^3| \leq 1 \Rightarrow \boxed{|(D_u f)(0,0)| \leq 1}$.

(c) $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$, $\gamma(0) = (0,0)$, $|\gamma'(0)| > 0$. $g(t) = f(\gamma(t))$
 Let $\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}$ Then $f(\gamma(t)) = \frac{(\gamma_1(t))^3}{(\gamma_1(t))^2 + (\gamma_2(t))^2}$ if $\gamma_1(t) \neq 0$ and $\gamma_2(t) \neq 0$ } not zero at the same time.

If $\gamma_1(t)$ and $\gamma_2(t) \neq 0$ at the same time, then since γ differentiable, $\gamma_1'(t)$ and $\gamma_2'(t)$ exists $\Rightarrow (f(\gamma(t)))'$ exists \Rightarrow differentiable.

If $\gamma_1(t) = 0$ and $\gamma_2(t) = 0$, $f(\gamma(t)) = 0$

$$\lim_{h \rightarrow 0} \frac{f\left(\begin{bmatrix} \gamma_1(t+h) \\ \gamma_2(t) \end{bmatrix}\right) - f\left(\begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}\right)}{h} = \lim_{h \rightarrow 0} \frac{\gamma_1(t+h)^3}{(\gamma_1(t+h))^2 + (\gamma_2(t+h))^2} = 0 = \lim_{h \rightarrow 0} \frac{\gamma_1(t+h)^3}{h^3} = \frac{\gamma_1'(t+h)^3}{h^2} - \frac{\gamma_2'(t+h)^2}{h^2}$$

not part of

$$\lim_{h \rightarrow 0} \frac{\frac{\delta_1(t+h)^3}{h^3}}{\frac{\delta_1(t+h)^2}{h^2} - \frac{\delta_2(t+h)^2}{h^2}} = \lim_{h \rightarrow 0} \frac{\frac{(\delta_1(t+h) - \delta_1(t))^3}{h^3}}{\frac{(\delta_1(t+h) - \delta_1(t))^2}{h^2} - \frac{(\delta_2(t+h) - \delta_2(t))^2}{h^2}}$$

$$= \frac{(\delta_1'(t))^3}{(\delta_1'(t))^2 + (\delta_2'(t))^2} \Rightarrow \text{differentiable}$$

Hence, $g = f \circ \gamma$ is differentiable as desired.

Originally, $f(\gamma(t)) = \frac{(\delta_1(t))^3}{(\delta_1(t))^2 + (\delta_2(t))^2} \Rightarrow (f(\gamma(t)))' = \frac{(\delta_1(t)^2 + \delta_2(t)^2) 3\delta_1'(t)\delta_1(t) - \delta_1(t)^3(2\delta_1'(t)\delta_1'(t) + 2\delta_2'(t)\delta_2'(t))}{(\delta_1(t)^2 + \delta_2(t)^2)^2}$

$$= \frac{(\delta_1'(t))^4 \delta_1'(t) + 3\delta_1(t)^2 \delta_2(t)^2 \delta_1'(t) - 2\delta_1(t)^3 \delta_2'(t) \delta_2'(t)}{(\delta_1(t)^2 + \delta_2(t)^2)^2}$$

To check continuous differentiability at $t+h$, where $\gamma(t) = \begin{bmatrix} \delta_1(t) \\ \delta_2(t) \end{bmatrix}$.

$$\lim_{h \rightarrow 0} \frac{f(\gamma(t+h))' - f(\gamma(t))'}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(\delta_1'(t+h))^4 \delta_1'(t+h) + 3\delta_1'(t)^2 \delta_2'(t)^2 \delta_1'(t+h) - 2\delta_1'(t)^3 \delta_2'(t) \delta_2'(t+h)}{(\delta_1'(t+h)^2 + \delta_2'(t+h)^2)^2} - \frac{\delta_1'(t)^4 \delta_1'(t) + 3\delta_1(t)^2 \delta_2(t)^2 \delta_1'(t) - 2\delta_1(t)^3 \delta_2'(t) \delta_2'(t)}{(\delta_1'(t)^2 + \delta_2'(t)^2)^2} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(\delta_1'(t))^4 \delta_1'(t+h) - \delta_1'(t)^5}{(\delta_1'(t+h)^2 + \delta_2'(t+h)^2)^2} + \frac{(\delta_1'(t)^2 \delta_2'(t)^2 \delta_1'(t+h) - \delta_1'(t)^3 \delta_2'(t)^2)}{(\delta_1'(t+h)^2 + \delta_2'(t+h)^2)^2} + \frac{(2\delta_1'(t)^2 \delta_2'(t)^2 \delta_1'(t+h) - 2\delta_1'(t)^3 \delta_2'(t) \delta_2'(t+h))}{(\delta_1'(t+h)^2 + \delta_2'(t+h)^2)^2} \right]$$

$$\rightarrow \lim_{h \rightarrow 0} \frac{1}{(\delta_1'(t)^2 + \delta_2'(t)^2)^2} \left[(\delta_1'(t))^4 \delta_1''(t) + \delta_1'(t)^2 \delta_2'(t)^2 \delta_1''(t) + 2\delta_1'(t)^2 \delta_2'(t)^2 \delta_1''(t) - 2\delta_1'(t)^3 \delta_2'(t) \delta_2''(t) \right]$$

= itself

Hence if γ continuously differentiable, g is continuously differentiable

3(d) Consider the path $(h, h) \rightarrow 0$.

$$\lim_{h \rightarrow 0} \frac{f(h, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^3}{h^2 + h^2} = \frac{1}{2} \neq 0. \text{ Actually, the}$$

$$\lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = 0$$

Hence f is not differentiable at $(0, 0)$.

Prgh 514 (A) No. Consider the matrix

$$G_\epsilon = \begin{bmatrix} 1 & \epsilon & & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

for $\epsilon > 0$

It is invertible.

This matrix is in its Jordan canonical form and hence cannot be diagonalizable. Note the identity matrix is diagonalizable. Take $\epsilon \rightarrow 0$, we can make the distance between G_ϵ and I_n to be arbitrarily small \Rightarrow set of diagonalizable matrix is not open in M_n .

(b) Consider the matrix $G_n = \begin{bmatrix} 1 & \epsilon & & 0 \\ & 1 + \frac{\epsilon}{n} & & \\ & & \ddots & \\ & & & 1 + \frac{\epsilon}{n-1} \\ 0 & & & & 1 \end{bmatrix}$

Note that this matrix G_n has n distinct eigenvalues \Rightarrow diagonalizable.

However $\lim_{n \rightarrow \infty} G_n = G_\epsilon$ which is not diagonalizable. Since the limit point is

not contained, the set of diagonalizable matrix is not closed.

(c) I claim that the set of diagonalizable matrix is dense in M_n .

Let A be a n -diagonalizable matrix. Then $A = P^{-1}JP$ for some matrix in Jordan Canonical form J . Let $\epsilon > 0$. Consider the following matrix:

$$\Lambda = \begin{bmatrix} \epsilon & & & \\ \frac{\epsilon}{2^{1/2} |P_{11}| |P_{12}|} & & & \\ & 2\epsilon & & \\ & \frac{\epsilon}{2^{1/2} |P_{11}| |P_{12}|} & & \\ & & \ddots & \\ & & & n\epsilon & \\ & & & \frac{\epsilon}{2^{1/2} |P_{11}| |P_{12}|} & \end{bmatrix}$$

Then $J + \Lambda \frac{1}{k}$ is diagonalizable for some $k \in \mathbb{Z}$.

This is because Λ perturbs all the diagonal elements and ensures $J + \Lambda \frac{1}{k}$ has distinct diagonal elements \Rightarrow distinct eigenvalues. } next page.

Then: Note $\|P^{-1}(J+\Lambda)P - A\|$
 $= \|P^{-1}(J+\Lambda)P - P^{-1}JP\|$
 $= \|P^{-1}(J+\Lambda - J)P\| = \|P^{-1}\Lambda P\|$
 $\leq \|P^{-1}\| \|\Lambda\| \|P\|$
 $< \epsilon.$

Hence, \exists diagonalizable matrix within distance ϵ from any non diagonalizable matrix A $\forall \epsilon$. Since ϵ arbitrary, the set of diagonalizable $n \times n$ matrices is dense in M_n .

Prob S. 24

$$f: \mathbb{R}^2 \rightarrow \mathbb{R} \quad f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Consider $\frac{\partial f}{\partial x}$ at $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left(\frac{xy(x^2-y^2)}{x^2+y^2} \right) = \frac{(x^2+y^2)(3x^2y - y^3) - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

Consider $\frac{\partial f}{\partial x}$ at $(x,y) = (0,0)$

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0$$

Clearly, if $(x,y) \neq (0,0)$, $\frac{\partial^2 f}{\partial x^2}$ exists since both numerator, denominator differentiable and denominator $\neq 0$. At $(0,0)$,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0-0}{h} = 0 \quad \therefore \boxed{\frac{\partial^2 f}{\partial x^2} \text{ exists everywhere}}$$

Note, x and y are symmetric up to a sign $\Rightarrow \boxed{\frac{\partial f}{\partial y^2} \text{ exists everywhere}}$

At $(x,y) \neq 0$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(\frac{xy(x^2-y^2)}{x^2+y^2} \right) = \frac{(x^2+y^2)(x^3 - 3xy^2) - (2y)xy(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

It is clear that $\frac{\partial f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial y}$ exist everywhere less the origin, since it's by quotient rule. Suffices to check at the origin that it exists

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h \rightarrow 0} \frac{-\frac{h^5}{h^4} - 0}{h} = \boxed{-1}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \lim_{h \rightarrow 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{h^5}{h^4} - 0}{h} = \boxed{1}$$

$\therefore \boxed{\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x} \text{ exists everywhere}}$ but $\boxed{\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x} \text{ at } (0,0)}$ as desired