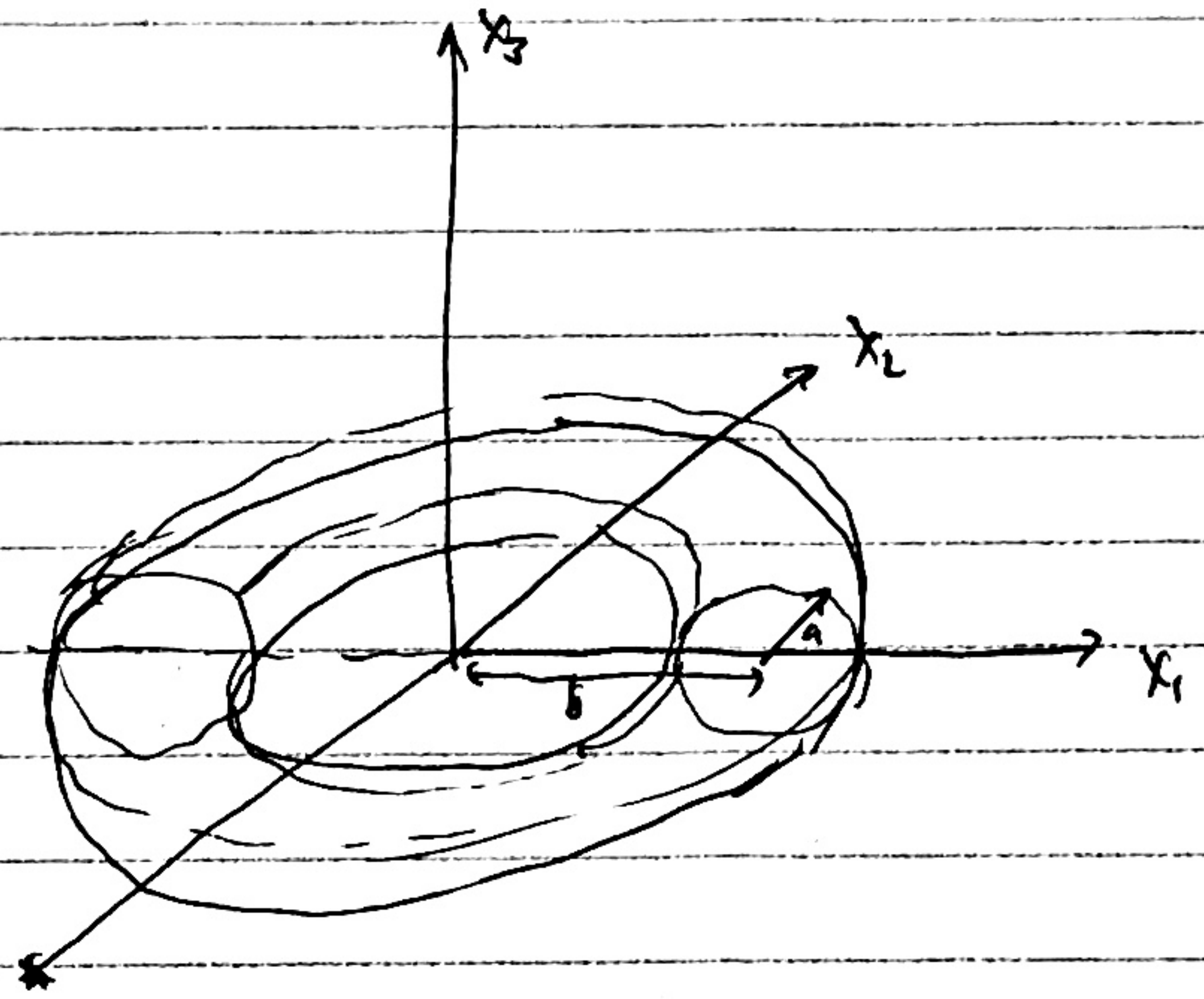
MATH 105 HW 10

Rudin 9.12 Let $0 < a < b$ and $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^3$

$f_1(s,t) = (b + a\cos s)\cos t$
$f_2(s,t) = (b + a\cos s)\sin t$
$f_3(s,t) = a\sin s$

From the expressions, $f_1^2 + f_2^2 = (b + a\cos s)^2 \Rightarrow \left(\sqrt{f_1^2+f_2^2} - b\right)^2 = (a\cos s)^2$

$$\Rightarrow \left(\sqrt{f_1^2+f_2^2} - b\right)^2 + f_3^2 = a^2$$



It describes a donut in 3D with diameter 2b (i.e. inner ring $b-a$ and outer ring $b+a$) and width diameter 2a.

a) $(\nabla f_1) = \begin{bmatrix} \frac{\partial f_1}{\partial s} \\ \frac{\partial f_1}{\partial t} \end{bmatrix} = \begin{bmatrix} \cancel{(b - a\sin s)\cos} - a\sin s\cos t \\ -(b + a\cos s)\sin t \end{bmatrix}$ For $\nabla f_1 = 0$, $-a\sin s \cos t = 0$ and $-(b + a\cos s)\sin t = 0$.

If $\cos t = 0$, then $\sin t \neq 0 \Rightarrow b + a\cos s = 0 \Rightarrow \cos s = -\frac{b}{a}$ (contradiction since $b > a \Rightarrow |\cos s| > 1$)

$\therefore \sin s = 0 \Rightarrow \sin t = 0$.

Hence, $f^{-1}(p) = \begin{bmatrix} s \\ t \end{bmatrix}$ s.t. $\sin s = 0$, $\sin t = 0$. $\Rightarrow f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} (b + a\cos s)\cos t \\ 0 \\ 0 \end{bmatrix}$

Since $\sin s = 0$, $\cos s = \pm 1$. Similarly, $\cos t = \pm 1$

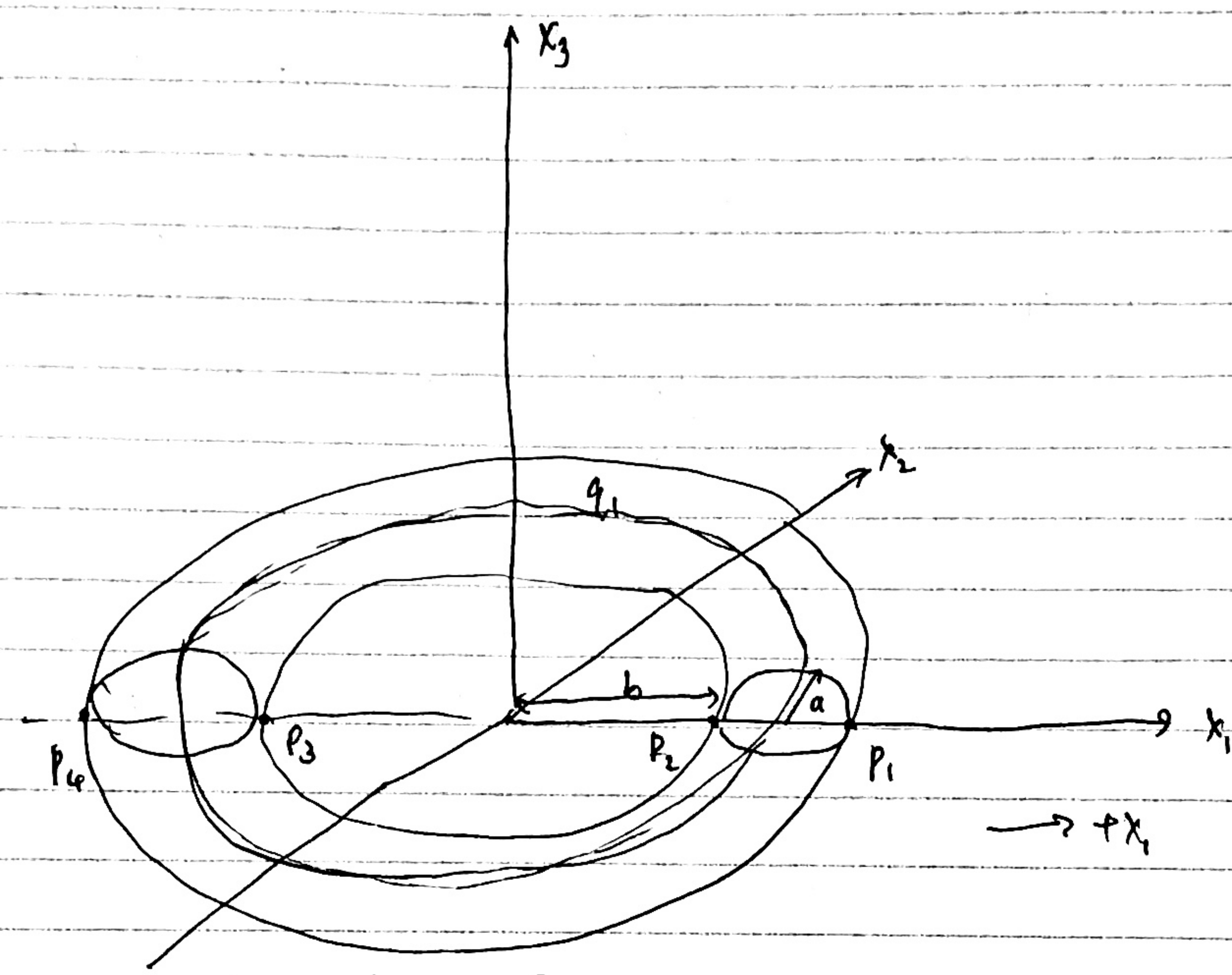
$\Rightarrow f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right)$ takes on 4 possible values $\begin{bmatrix} b-a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -a-b \\ 0 \\ 0 \end{bmatrix}$.

$\therefore \boxed{p \in \left\{ \begin{bmatrix} b-a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a-b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -a-b \\ 0 \\ 0 \end{bmatrix} \right\}.}$

b) $(\nabla f_3) = \begin{bmatrix} \frac{\partial f_3}{\partial s} \\ \frac{\partial f_3}{\partial t} \end{bmatrix} = \begin{bmatrix} a\cos s \\ 0 \end{bmatrix} \Rightarrow a\cos s = 0 \Rightarrow s = \frac{1}{2}\pi + k\pi, k \in \mathbb{Z}$

$\therefore f^{-1}(q) = \begin{bmatrix} \frac{1}{2}\pi + k\pi \\ t \end{bmatrix}$ for some $k \in \mathbb{Z}$, $t \in \mathbb{R}$ $\Rightarrow q \in \left\{ \begin{bmatrix} b\cos t \\ b\sin t \\ a \end{bmatrix}, \begin{bmatrix} b\cos t \\ b\sin t \\ -a \end{bmatrix} \right\}_{t \in \mathbb{R}}$

1 (c)



Consider $\nabla^2 f_1 = \begin{bmatrix} \frac{\partial^2 f_1}{\partial s^2} & \frac{\partial^2 f_1}{\partial s \partial t} \\ \frac{\partial^2 f_1}{\partial t \partial s} & \frac{\partial^2 f_1}{\partial t^2} \end{bmatrix} = \begin{bmatrix} -a\cos s \cos t & a \sin s \sin t \\ a \sin s \sin t & -(b+a\cos s)\cos t \end{bmatrix}$

For maximum point, $\nabla^2 f_1$ must be negative semi definite, i.e.

$-a\cos s \cos t \leq 0$

$\det \nabla^2 f_1 = (-a\cos s \cos t)(-(b+a\cos s)\cos t) - (a \sin s \sin t)^2 \geqslant 0$] by Sylvester's rule of inertia

$\Rightarrow (\cos s, \cos t) = (1,1)$ (only combination that satisfies this) $\Rightarrow \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}$ is maximum

Similarly, for minimum point, $\nabla^2 f_1$ must be positive semidefinite i.e.

$-a\cos s \cos t \geqslant 0$ by Sylvester's rule of inertia.

$\det \nabla^2 f_1 \geqslant 0$

Only $(\cos s, \cos t) = (1, -1)$ satisfies this $\Rightarrow \begin{bmatrix} -a-b \\ 0 \\ 0 \end{bmatrix}$ is minimum.

Geometrically, it is also very easy to tell that these two points are indeed the maximum and minimum of the x_1 components.

$\nabla^2 f_3 = \begin{bmatrix} -a\sin s & 0 \\ 0 & 0 \end{bmatrix}$ We already know $\sin s = \pm 1$ (since $\cos s = 0$)

Hence, if $\sin s = 1$, $\nabla^2 f_3$ is negative definite $\Rightarrow$ maximum point

if $\sin s = -1$, $\nabla^2 f_3$ is positive definite $\Rightarrow$ minimum point.

1(c) continued

Hence all points $\left\{\begin{bmatrix} b\cos t \\ b\sin t \\ a \end{bmatrix} : t \in \mathbb{R}\right\}$ are maximum points

and all points $\left\{\begin{bmatrix} b\cos t \\ b\sin t \\ -a \end{bmatrix} : t \in \mathbb{R}\right\}$ are minimum points

Geometrically, the first set is the fillings of donut (upper filling) while the second set is the lower filling.

Rudin 9.13

$f : \mathbb{R} \to \mathbb{R}^3$. Let $f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$ $\quad |f(t)| = 1 \Rightarrow \sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2} = 1$

Then $f'(t) = \begin{bmatrix} \frac{d}{dt} f_1(t) \\ \frac{df_2}{dt}(t) \\ \frac{df_3}{dt}(t) \end{bmatrix}$

$$\Rightarrow \frac{d}{dt}\left(\sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2}\right) = 0$$

$$\Rightarrow \frac{1}{2\sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2}}\left(2f_1(t)f_1'(t) + 2f_2(t)f_2'(t) + 2f_3(t)f_3'(t)\right) = 0$$

$$\Rightarrow f_1(t)f_1'(t) + f_2(t)f_2'(t) + f_3(t)f_3'(t) = 0$$

$$\therefore f'(t)\cdot f(t) = \frac{d}{dt}f_1(t)\cdot f_1(t) + \frac{df_2}{dt}(t)\, f_2(t) + \frac{df_3}{dt}(t)\, f_3(t)$$

$$= f_1'(t)f_1(t) + f_2'(t)f_2(t) + f_3'(t)f_3(t) = 0 \text{ as desired.}$$

Rudin 9.14

$f(0,0) = 0$; $f(x,y) = \frac{x^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$

(a) At $(x,y) \neq (0,0)$, $D_1 f = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}\right) = \frac{(x^2+y^2)3x^2 - x^3\cdot 2x}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}$

At $(x,y) = (0,0)$, $\left.\frac{\partial f}{\partial x}\right|_{p=(0,0)} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{f(h,0)}{h} = \lim_{h\to 0}\frac{\frac{h^3}{h^2}}{h} = 1$

Note $x^4 + 3x^2y^2 < 2x^4 + 4x^2y^2 + 2y^4 = 2(x^2+y^2)^2$

$$\rightarrow \left|\frac{x^4+3x^2y^2}{(x^2+y^2)^2}\right| < \frac{2x^4+4x^2y^2+2y^4}{(x^2+y^2)^2} = 2 \quad \text{Hence } \boxed{D_1 f \text{ is bounded.}}$$

(b) At $(x,y) \neq (0,0)$, $D_2 f = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3}{x^2+y^2}\right) = \frac{\cancel{(x^2+y^2)} - x^3 2y}{(x^2+y^2)^2} = -\frac{2x^3y}{(x^2+y^2)^2}$

At $(x,y) = (0,0)$, $\left.\frac{\partial f}{\partial y}\right|_{p=(0,0)} = \lim_{h\to 0}\frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$ } next page.

Note $\left|\frac{2x^3y}{(x^2+y^2)^2}\right| < 1$. This is because $x^4 + 2x^2y^2 \overset{\text{AM-GM}}{\geq} 2\sqrt{x^4x^2y^2}$
$= 2|x^3y|$

Hence $\left|\frac{2x^3y}{(x^2+y^2)^2}\right| < 1 \Rightarrow$ $\boxed{D_2 f \text{ is bounded.}}$

f is therefore continuous because for $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M_1 + M_2}$ where M_1 and M_2 are the bounds above.

Then $d((x_1, y_1), (x_2, y_2)) < \delta$

$\Rightarrow |f(x_1, y_1) - f(x_2, y_2)| = |f(x_1, y_1) - f(x_2, y_1) + f(x_2, y_1) - f(x_2, y_2)|$

$\overset{\Delta}{\leq} |f(x_1, y_1) - f(x_2, y_1)| + |f(x_2, y_1) - f(x_2, y_2)|$

$\overset{\text{Mean value}}{<} \delta \cdot M_1 + \delta M_2 = \delta(M_1 + M_2) = \frac{\varepsilon}{M_1 + M_2}(M_1 + M_2) = \varepsilon \Rightarrow$ f is continuous as desired.

3 (b) Let $u = \begin{bmatrix} a \\ b \end{bmatrix}$. Then $D_u f(0,0) = \lim_{h\to 0} \frac{f(ha, hb) - f(0,0)}{h}$
with $a^2+b^2=1$

$= \lim_{h\to 0} \frac{\frac{h^3a^3}{h^2a^2+h^2b^2} - 0}{h} = a^3$ (since $a^2+b^2=1$)
$\therefore D_u f(0,0)$ exists.

Since $a^2+b^2=1$, $a^2 \leq 1 \Rightarrow |a| \leq 1 \Rightarrow |a^3| \leq 1 \Rightarrow \boxed{|(D_u f)(0,0)| \leq 1.}$

(c) $\gamma: \mathbb{R} \to \mathbb{R}^2$, $\gamma(0) = (0,0)$, $|\gamma'(0)| > 0$. $g(t) = f(\gamma(t))$
Let $\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}$ Then $f(\gamma(t)) = \frac{(\gamma_1(t))^3}{(\gamma_1(t))^2 + (\gamma_2(t))^2}$ if $\gamma_1(t) \neq 0$ and $\gamma_2(t) \neq 0$] not zero at the same time.

If $\gamma_1(t)$ and $\gamma_2(t) \neq 0$ at the same time, then since γ differentiable, $\gamma_1'(t)$ and $\gamma_2'(t)$ exists $\Rightarrow$ $(f(\gamma(t)))'$ exists $\Rightarrow$ differentiable.

If $\gamma_1(t) = 0$ and $\gamma_2(t) = 0$, $f(\gamma(t)) = 0$

$\lim_{h\to 0} \frac{f\left(\begin{bmatrix} \gamma_1(t+h) \\ \gamma_2(t) \end{bmatrix}\right) - f\left(\begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix}\right)}{h} = \lim_{h\to 0} \frac{\frac{\gamma_1(t+h)^3}{(\gamma_1(t+h))^2 + (\gamma_2(t+h))^2} - 0}{h} = \lim_{h\to 0} \frac{\frac{\gamma_1(t+h)^3}{h^3}}{\frac{\gamma_1(t+h)^2}{h^2} - \frac{\gamma_2(t+h)^2}{h^2}}$

next page

$$\lim_{h\to 0} \frac{\frac{\gamma_1(t+h)^3}{h^3}}{\frac{\gamma_1(t+h)^2}{h^2} - \frac{\gamma_2(t+h)^2}{h^2}} = \lim_{h\to 0} \frac{\frac{(\gamma_1(t+h)-\gamma_1(t))^3}{h^3}}{\frac{(\gamma_1(t+h)-\gamma_1(t))^2}{h^2} - \frac{(\gamma_2(t+h)-\gamma_2(t))^2}{h^2}}$$

$$= \frac{(\gamma_1'(t))^3}{(\gamma_1'(t))^2 + (\gamma_2'(t))^2} \Rightarrow \text{differentiable}$$

Hence, $\boxed{g = f \circ \gamma \text{ is differentiable}}$ as desired.

Originally, $f(\gamma(t)) = \frac{(\gamma_1(t))^3}{(\gamma_1(t))^2 + (\gamma_2(t))^2} \Rightarrow (f(\gamma(t)))' = \frac{(\gamma_1(t)^2 + \gamma_2(t)^2)\, 3\gamma_1^2(t)\gamma_1'(t) - \gamma_1(t)^3 (2\gamma_1(t)\gamma_1'(t) + 2\gamma_2(t)\gamma_2'(t))}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2}$

$$= \frac{(\gamma_1(t))^4\gamma_1'(t) + 3\gamma_1(t)^2\gamma_2(t)^2\gamma_1'(t) - 2\gamma_1(t)^3\gamma_2(t)\gamma_2'(t)}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2}$$

To check continuous differentiability at $t+h$, where $\gamma(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\lim_{h\to 0} \frac{f(\gamma(t+h))' - f(\gamma(t))'}{h} = \lim_{h\to 0} \frac{1}{h} \left[\frac{(\gamma_1'(t))^4\gamma_1'(t+h) + 3\gamma_1'(t)^2\gamma_2'(t)^2\gamma_1'(t+h) - 2\gamma_1(t)'^3\gamma_2'(t)\gamma_2'(t+h)}{((\gamma_1'(t))^2 + (\gamma_2'(t))^2)^2} - \frac{\gamma_1'(t)^3}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{\left[(\gamma_1'(t))^4\gamma_1'(t+h) - \gamma_1'(t)^5\right] + \left[\gamma_1'(t)^2\gamma_2'(t)^2\gamma_1'(t+h) - \gamma_1'(t)^3\gamma_2'(t)^2\right] + \left(2\gamma_1'(t)^2\gamma_2'(t)^2\gamma_1'(t+h) - 2\gamma_1'(t)^3\gamma_2(t)\gamma_2'(t+h)\right)}{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}$$

$$= \lim_{h\to 0} \frac{1}{(\gamma_1'(t)^2 + \gamma_2'(t)^2)} \left[(\gamma_1'(t))^4\gamma_1''(t) + \gamma_1'(t)^2\gamma_2'(t)\gamma_1''(t) + 2\gamma_1'(t)^2\gamma_2'(t)^2\gamma_1''(t) - 2\gamma_1'(t)^3\gamma_2(t)\gamma_2''(t) \right]$$

= itself

~~Hence if γ continuously differentiable, g is continuously differentiable~~

3(d) Consider the path $(h,h) \to 0$.

$$\lim_{h\to 0} \frac{f(h,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2+h^2} - 0}{h} = \frac{1}{2} \neq 0.$$ ~~Actually, the~~

$$\lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0$$

Hence $\boxed{f \text{ is not differentiable at } (0,0)}$.

Pugh 5.14 (a) No. Consider the matrix

$$G_\varepsilon = \begin{bmatrix} 1 & \varepsilon & & & 0 \\ 0 & 1 & & & \\ \vdots & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

for $\varepsilon > 0$. It is invertible.

This matrix is in its Jordan canonical form and hence cannot be diagonalizable. Note the identity matrix is diagonalizable. Take $\varepsilon \to 0$, we can make the distance between G_ε and I_n to be arbitrarily small $\Rightarrow$ set of diagonalizable matrix is $\boxed{\text{not open}}$ in M_n.

b) Consider the matrix $G_n =$

$$G_n = \begin{bmatrix} 1 & \varepsilon & & & & 0 \\ & 1+\frac{\varepsilon}{n} & & & & \\ & & 1+\frac{\varepsilon}{n+1} & & & \\ & & & 1+\frac{\varepsilon}{n+2} & & \\ & & & & \ddots & \\ 0 & & & & & 1+\frac{\varepsilon}{2n-1} \end{bmatrix}$$

Note that this matrix G_n has n distinct eigenvalues $\Rightarrow$ diagonalizable. ~~G thn~~.
However $\lim_{n\to\infty} G_n = G_\varepsilon$ which is not diagonalizable. Since the limit point is not contained, the set of diagonalizable matrix is $\boxed{\text{not closed}}$.

c) I claim that the set of diagonalizable matrix is dense in M_n.
Let A be a non-diagonalizable matrix. Then $A = P^{-1}JP$ for some matrix in Jordan Canonical Form J. Let $\varepsilon > 0$. Consider the following matrix:

$$\Lambda = \begin{bmatrix} \frac{\varepsilon}{2^n\|P\|\|P^{-1}\|} & & \\ & \frac{2\varepsilon}{2^n\|P\|\|P^{-1}\|} & \\ & & \ddots \\ & & & \frac{n\varepsilon}{2^n\|P\|\|P^{-1}\|} \end{bmatrix}$$

Then $J + \Lambda\frac{1}{k}$ is diagonalizable for some $k \in \mathbb{Z}$. This is because Λ perturbs all the diagonal elements and ensures $J + \Lambda\frac{1}{k}$ has distinct diagonal elements $\Rightarrow$ distinct eigenvalues] next page.

Then: Note $\| P^{-1}(J+\Lambda)P - A\|$
$= \| P^{-1}(J+\Lambda)P - P^{-1}JP\|$
$= \| P^{-1}(J+\Lambda - J)P\| = \|P^{-1}\Lambda P\|$
$\leq \|P^{-1}\|\|\Lambda\|\|P\|$
$< \varepsilon.$

Hence, $\exists$ diagonalizable matrix within distance ε from any non diagonalizable matrix A $\forall \varepsilon$. Since ε arbitrary, the set of diagonalizable $n\times n$ matrices is $\boxed{\text{dense}}$ in M_n.

Pugh 5.24 $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Consider $\frac{\partial f}{\partial x}$ at $(x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = \frac{(x^2+y^2)(3x^2y-y^3) - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

Consider $\frac{\partial f}{\partial x}$ at $(x,y) = (0,0)$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

Clearly, if $(x,y) \neq (0,0)$, $\frac{\partial^2 f}{\partial x^2}$ exists since both numerator, denominator differentiable and denominator $\neq 0$. At $(0,0)$,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = \boxed{0}$$

$\therefore$ $\boxed{\frac{\partial^2 f}{\partial x^2} \text{ exists everywhere}}$

Note, x and y are symmetric up to a sign $\Rightarrow$ $\boxed{\frac{\partial^2 f}{\partial y^2} \text{ exists everywhere.}}$

At $(x,y) \neq 0$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) = \frac{(x^2+y^2)(x^3-3xy^2) - (2y)xy(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$$

It is clear that $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial y}$ exist everywhere less the origin, since it's by quotient rule.
Suffices to check at the origin that it exists

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h\to 0}\frac{-\frac{h^5}{h^4}-0}{h} = \boxed{-1}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{h\to 0}\frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = \lim_{h\to 0}\frac{\frac{h^5}{h^4}-0}{h} = \boxed{1}$$

$\therefore$ $\boxed{\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x} \text{ exists everywhere}}$ but $\boxed{\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x} \text{ at } (0,0)}$ as desired

Rudin 9.19

$$3x + y - z + u^2 = 0 \quad \text{—(1)}$$
$$x - y + 2z + u = 0 \quad \text{—(2)}$$
$$2x + 2y - 3z + 2u = 0 \quad \text{—(3)}$$

$(1) - (2) - (3) \Rightarrow u^2 - u - 2u = 0 \Rightarrow u = 0$ or 3

consider $f(x,y,z,u) = \begin{bmatrix} 3x + y - z + u^2 \\ x - y + 2z + u \\ 2x + 2y - 3z + 2u \end{bmatrix}$

Then, $\nabla f = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}$

If u is the unknown, $\begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 0 \end{bmatrix}$ ⇒ determinant $= 0$
⇒ not invertible
⇒ cannot solve for u in terms of x, y, z.

If x is the unknown, $\begin{bmatrix} \cancel{3} & 1 & -1 & 2u \\ \cancel{1} & -1 & 2 & 1 \\ \cdot & 2 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2u+1 \\ 0 & 1 & 4 \end{bmatrix}$ when $u = 0, 3$, both are invertible
∴ can solve for x in terms of u, y, z

If y is the unknown $\begin{bmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & 0 \\ 0 & -7 & 2u-3 \end{bmatrix}$ When $u = 0, 3$, both are invertible
⇒ can solve for ~~y~~ in terms of ~~x~~, z, u

If z is the unknown, $\begin{bmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & 0 \\ 0 & 4 & 2u-3 \end{bmatrix}$ when $u = 0, 3$, both are invertible
⇒ can solve for z in terms of x, y, u.