WANG JIANZHI 04/13/2022 Page 1

MATH 105 HW 11

1(a) $\Omega_1 = |x|^{-2}(x_1\,dx_2 - x_2\,dx_1)$

$$= \frac{1}{x_1^2+x_2^2}(x_1\,dx_2 - x_2\,dx_1) = \frac{x_1}{x_1^2+x_2^2}\,dx_2 - \frac{x_2}{x_1^2+x_2^2}\,dx_1$$

$$\Rightarrow d\Omega_1 = \frac{\partial}{\partial x_1}\left(\frac{x_1}{x_1^2+x_2^2}\right)dx_1\wedge dx_2 - \frac{\partial}{\partial x_2}\left(\frac{x_2}{x_1^2+x_2^2}\right)dx_2\wedge dx_1$$

$$= \frac{(x_1^2+x_2^2)\cdot 1 - x_1(2x_1)}{(x_1^2+x_2^2)^2}\,dx_1\wedge dx_2 - \frac{(x_1^2+x_2^2)\cdot 1 - x_2(2x_2)}{(x_1^2+x_2^2)^2}\,dx_2\wedge dx_1$$

$$= \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}\,dx_1\wedge dx_2 - \frac{x_1^2-x_2^2}{(x_1^2+x_2^2)^2}\,dx_2\wedge dx_1$$

$$= dx_1\wedge dx_2\left(\frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2} - \frac{x_2^2-x_1^2}{(x_1^2+x_2^2)^2}\right) = \boxed{0}$$ as desired.

1(b) $\Omega_2 = |x|^{-3}(x_1\,dx_2\wedge dx_3 - x_2\,dx_1\wedge dx_3 + x_3\,dx_1\wedge dx_2)$

$$= |x|^{-3}(x_1\,dx_2\wedge dx_3 + x_2\,dx_3\wedge dx_1 + x_3\,dx_1\wedge dx_2)$$

$$= \frac{x_1}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}\,dx_2\wedge dx_3 + \frac{x_2}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}\,dx_3\wedge dx_1 + \frac{x_3}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}\,dx_1\wedge dx_2$$

$$d\Omega_2 = \frac{\partial}{\partial x_1}\left(\frac{x_1}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}\right)dx_1\wedge dx_2\wedge dx_3 + \frac{\partial}{\partial x_2}\left(\frac{x_2}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}\right)dx_2\wedge dx_3\wedge dx_1$$

$$+ \frac{\partial}{\partial x_3}\left(\frac{x_3}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}}\right)dx_3\wedge dx_1\wedge dx_2$$

$$= dx_1\wedge dx_2\wedge dx_3\left(\frac{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}\cdot 3 - \sum x_i\cdot\frac{3}{2}(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}\cdot 2x_i}{(x_1^2+x_2^2+x_3^2)^3}\right)$$

$$= dx_1\wedge dx_2\wedge dx_3\,\frac{3(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}} - 3(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}\cdot(x_1^2+x_2^2+x_3^2)}{(x_1^2+x_2^2+x_3^2)^3} = \boxed{0}$$

as desired

1(c) For the general n, $\boxed{\Omega_n = |x|^{-(n+1)}\left(\sum_{i=1}^{n+1} x_i\cdot(-1)^{i+1}\,dx_1\wedge dx_2\wedge\cdots\wedge\widehat{dx_i}\wedge dx_{i+1}\wedge\cdots\wedge dx_{n+1}\right)}$

I claim that $d\Omega_n = 0$.

$$\Omega_n = |x|^{-(n+1)} \left(\sum_{i=1}^{n+1} x_i \, (-1)^{i+1} \, dx_1 \wedge \dots \wedge dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \wedge \dots \wedge dx_{n+1} \right)$$

$$\Rightarrow d\Omega_n = \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|^{n+1}}\right)(-1)^{i+1} dx_i \wedge dx_1 \wedge \dots \wedge dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \wedge \dots \wedge dx_{n+1}$$

$$= \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|^{n+1}}\right) dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_i \wedge dx_{i+1} \wedge \dots \wedge dx_{n+1}$$

$$= dx_1 \wedge \dots \wedge dx_{n+1} \left(\sum_{i=1}^{n+1} \frac{\partial}{\partial x_i} \frac{x_i}{|x|^{n+1}} \right)$$

$$= dx_1 \wedge \dots \wedge dx_{n+1} \left(\sum_{i=1}^{n+1} \frac{|x|^{n+1} \cdot 1 - x_i \cdot \frac{n+1}{2} |x|^{n-1} \cdot 2x_i}{\left(|x|^{n+1}\right)^2} \right)$$

$$= dx_1 \wedge \dots \wedge dx_{n+1} \left(\frac{1}{\left(|x|^{n+1}\right)^2} \sum_{i=1}^{n+1} \left(|x|^{n+1} - (n+1)\, x_i^2 \, |x|^{n-1} \right) \right)$$

$$= dx_1 \wedge \dots \wedge dx_{n+1} \left(\frac{1}{\left(|x|^{n+1}\right)^2} \left((n+1)|x|^{n+1} - (n+1) \sum_{i=1}^{n+1} x_i^2 \, |x|^{n-1} \right) \right)$$

$$= dx_1 \wedge \dots \wedge dx_{n+1} \left(\frac{1}{\left(|x|^{n+1}\right)^2} \left((n+1)|x|^{n+1} - (n+1)|x|^{n+1} \right) \right) = \boxed{0}$$ as desired.

1(c) $\gamma(s,t) = \begin{bmatrix} \sin(\pi s)\cos(2\pi t) \\ \sin(\pi s)\sin(2\pi t) \\ \cos(\pi s) \end{bmatrix}$ $\quad 0 \le s,t \le 1.$

$$\int_\gamma \Omega_2 = \int_\gamma \frac{x_1}{|x|^3} dx_2 \wedge dx_3 - \frac{x_2}{|x|^3} dx_1 \wedge dx_3 + \frac{x_3}{|x|^3} dx_1 \wedge dx_2$$

$$= \int_\gamma x_1 \, dx_2 \wedge dx_3 - x_2 \, dx_1 \wedge dx_3 + x_3 \, dx_1 \wedge dx_2$$

$$= \int_{[0,1]^2} \left(\sin(\pi s)\cos(2\pi t) \begin{vmatrix} \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial t} \\ \frac{\partial x_3}{\partial s} & \frac{\partial x_3}{\partial t} \end{vmatrix} - (\sin \pi s)\sin(2\pi t) \begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial t} \\ \frac{\partial x_3}{\partial s} & \frac{\partial x_3}{\partial t} \end{vmatrix} + \cos(\pi s) \begin{vmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial s} & \frac{\partial x_2}{\partial t} \end{vmatrix} \right) ds\,dt$$

Note: $\frac{\partial x_1}{\partial s} = \cos(2\pi t)\, \pi \cos(\pi s)$ $\quad \frac{\partial x_2}{\partial s} = \sin(2\pi t)\, \pi \cos(\pi s)$ $\quad \frac{\partial x_3}{\partial s} = -\pi \sin \pi s$

$\frac{\partial x_1}{\partial t} = -\sin(\pi s)\, 2\pi \sin(2\pi t)$ $\quad \frac{\partial x_2}{\partial t} = 2\pi \sin(\pi s)\cos(2\pi t)$ $\quad \frac{\partial x_3}{\partial t} = 0.$

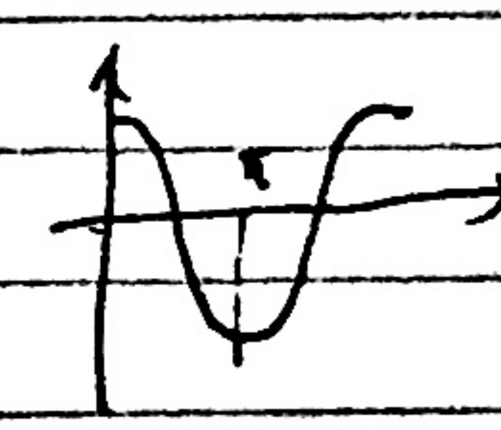
$$\therefore \int_\gamma \Omega_2 = \int_{[0,1]^2} \sin(\pi s)\cos(2\pi t)\left(-\pi\sin(\pi s)\cdot 2\pi\cdot\sin(\pi s)\cos(2\pi t)\right)$$

$$- \sin(\pi s)\sin(2\pi t)\left(-\pi\sin\pi s\cdot 2\pi\cdot\sin(\pi s)\sin(2\pi t)\right)$$

$$+ \cos(\pi s)\cdot\Big(\pi\cdot 2\pi\cdot\cos(2\pi t)\cos(\pi s)\sin(\pi s)\cos(2\pi t)$$

$$+ 2\pi\cdot\pi\sin(\pi s)\sin(2\pi t)\sin(2\pi t)\cos(\pi s)\Big)\, dt\, ds$$

$$= \int_{[0,1]^2}\left(2\pi^2\sin^3\pi s + 2\pi^2\cos^2\pi s\,\sin\pi s\right) dt\, ds$$



$$= \int_{[0,1]^2} 2\pi^2\sin\pi s\, dt\, ds = \int_0^1\left[\int_0^1 2\pi^2\sin\pi s\, ds\right] dt$$

$$= \int_0^1\left[-2\pi^2\frac{\cos\pi s}{\pi}\right]_0^1 dt = \int_0^1 4\pi\, dt$$

$$= \boxed{4\pi} \text{ as desired.}$$

1(f) $\gamma(a,b) = \begin{bmatrix} \frac{2a}{1+a^2+b^2} \\ \frac{2b}{1+a^2+b^2} \\ \frac{-1+a^2+b^2}{1+a^2+b^2} \end{bmatrix}$

$$\int_\gamma \Omega_2 = \int_{\mathbb{R}^2}\left(\frac{2a}{1+a^2+b^2}\cdot\begin{vmatrix}\frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} \\ \frac{\partial x_3}{\partial a} & \frac{\partial x_3}{\partial b}\end{vmatrix} - \frac{2b}{1+a^2+b^2}\begin{vmatrix}\frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} \\ \frac{\partial x_3}{\partial a} & \frac{\partial x_3}{\partial b}\end{vmatrix} + \frac{-1+a^2+b^2}{1+a^2+b^2}\begin{vmatrix}\frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b}\end{vmatrix}\right) da\, db$$

$$\frac{\partial x_1}{\partial a} = \frac{2-2a^2+2b^2}{(1+a^2+b^2)^2} \qquad \frac{\partial x_2}{\partial a} = -\frac{4ab}{(1+a^2+b^2)^2} \qquad \frac{\partial x_3}{\partial a} = \frac{4a}{(1+a^2+b^2)^2}$$

$$\frac{\partial x_1}{\partial b} = -\frac{4ab}{(1+a^2+b^2)^2} \qquad \frac{\partial x_2}{\partial b} = \frac{2-2b^2+2a^2}{(1+a^2+b^2)^2} \qquad \frac{\partial x_3}{\partial b} = \frac{4b}{(1+a^2+b^2)^2}$$

$$\Rightarrow \int_\gamma \Omega_2 = \int_{\mathbb{R}^2}\Bigg[\frac{2a}{1+a^2+b^2}\cdot\frac{-8a^3-8ab^2-8a}{(1+a^2+b^2)^4}$$

$$- \frac{2b}{1+a^2+b^2}\cdot\frac{8b+8a^2b+8b^3}{(1+a^2+b^2)^4}$$

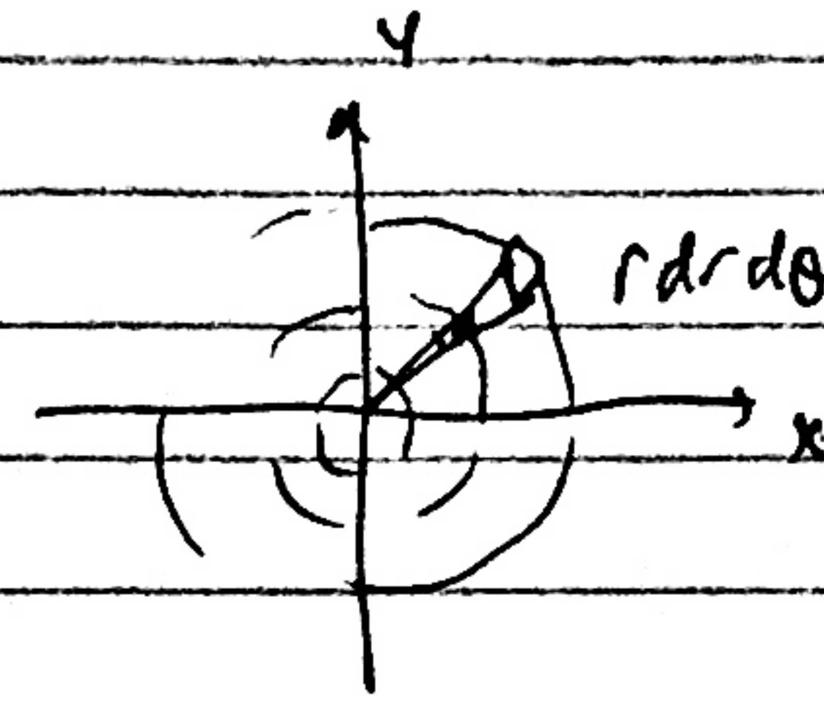
$$+ \frac{-1+a^2+b^2}{1+a^2+b^2}\cdot\frac{4-4a^4-4b^4-8a^2b^2}{(1+a^2+b^2)^4}\Bigg] da\, db$$

$$= \int_{\mathbb{R}^2}\frac{-4(a^6+b^6+1+3a^4+3b^4+3a^2+3b^2+6a^2b^2+3a^2b^4+3a^4b^2)}{(1+a^2+b^2)^5}\, da\, db$$

↳ next page

$$= \int_{\mathbb{R}^2} \frac{-4(a^2+b^2+1)^3}{(1+a^2+b^2)^5}\, da\, db$$

$$= \int_{\mathbb{R}^2} -\frac{4}{(1+a^2+b^2)^2}\, da\, db.$$



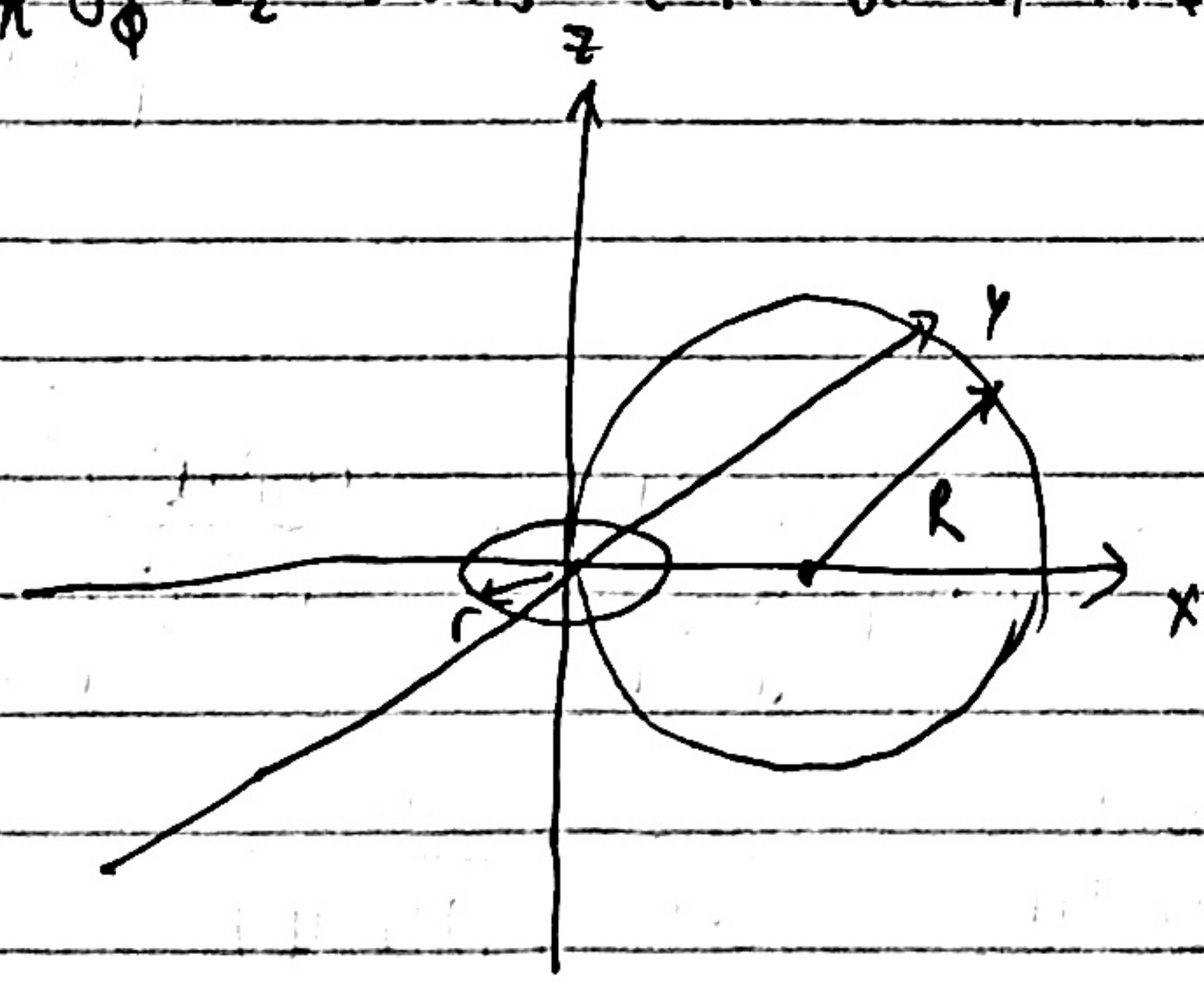
$$= \iint -\frac{4}{(1+r^2)^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\int_0^{\infty} -\frac{4r}{(1+r^2)^2}\, dr\right] d\theta = \int_0^{2\pi}\left[\cancel{\tfrac{4}{2}}\, 2\cdot(1+r^2)^{-1}\right]_0^{\infty} d\theta$$

$$= \int_0^{2\pi} -2\, d\theta = \boxed{-4\pi}$$

Not sure why there's a '−' sign.

2. $\frac{1}{4\pi}\int_\phi \Omega_2$ counts the number of times two curves loop around each other.



$$\gamma_1(\alpha) = \begin{bmatrix} r\cos(2\pi\alpha) \\ r\sin(2\pi\alpha) \\ 0 \end{bmatrix}$$

$$\gamma_2(\beta) = \begin{bmatrix} R + R\cos(2\pi\beta) \\ 0 \\ R\sin(2\pi\beta) \end{bmatrix}$$

$$\phi(\alpha,\beta) = \begin{bmatrix} r\cos(2\pi\alpha) - R - R\cos(2\pi\beta) \\ r\sin(2\pi\alpha) \\ -R\sin(2\pi\beta) \end{bmatrix}$$

$$\int_\phi \Omega_2 = \int_\phi \frac{x_1}{|x|^3}\, dx_2\wedge dx_3 - \frac{x_2}{|x|^3}\, dx_1\wedge dx_3 + \frac{x_3}{|x|^3}\, dx_1\wedge dx_2$$

$$|x|^2 = r^2 + 2R\left(R - 2r\cos(2\pi\alpha)\right)\left(1+\cos(2\pi\beta)\right)$$