MATH 105 HW 12.

1. I referenced math.purdue.edu/~arapura/preprints/diffforms.pdf.

Let ω be a 2-form on $\mathbb{R}^3$. Then, it can be expressed as
$\omega = f\,dx\wedge dy + g\,dz\wedge dx + h\,dz\wedge dy.$

Since ω is assumed to be closed, $d\omega = f_z\,dz\wedge dx\wedge dy + g_y\,dy\wedge dz\wedge dx + h_x\,dx\wedge dz\wedge dy$
$= (f_z + g_y - h_x)\,dx\wedge dy\wedge dz = 0$

$\Rightarrow f_z + g_y - h_x = 0 \Rightarrow \boxed{f_z = h_x - g_y}$

Define the following operation: (Purdue's notes called it homotopy) continuous deformation of functions

$$H_z(\omega) = \left(\int_0^z g(x,y,z)\,dz\right)dx + \left(\int_0^z h(x,y,z)\,dz\right)dy.$$

Then:

$$dH_z(\omega) = \left(\frac{d}{dy}\left(\int_0^z g(x,y,z)\,dz\right)\right)dy\wedge dx + \left(\frac{d}{dz}\left(\int_0^z g(x,y,z)\,dz\right)\right)dz\wedge dx$$
$$+\left(\frac{d}{dz}\left(\int_0^z h(x,y,z)\,dz\right)\right)dz\wedge dy + \left(\frac{d}{dx}\left(\int_0^z h(x,y,z)\,dz\right)\right)dx\wedge dy$$

$$= \left(\int_0^z \cancel{-}(g_y - h_x)\,dz\right)dx\wedge dy + \big(g(x,y,z) - g(x,y,0)\big)\,dz\wedge dx$$
$$+ \big(h(x,y,z) - h(x,y,0)\big)\,dz\wedge dy$$

(over here, we did many switching of integrations and differentiations)

Since $f_z = h_x - g_y$, this reduces to

$$= \left(\int_0^z f_z\,dz\right)dx\wedge dy + \big(g(x,y,z) - g(x,y,0)\big)\,dz\wedge dx$$
$$+ \big(h(x,y,z) - h(x,y,0)\big)\,dz\wedge dy$$

$$= \big(f(x,y,z) - f(x,y,0)\big)\,dx\wedge dy + \big(g(x,y,z) - g(x,y,0)\big)\,dz\wedge dx$$
$$+ \big(h(x,y,z) - h(x,y,0)\big)\,dz\wedge dy$$

$= \omega(x,y,z) - \omega(x,y,0)$ } next page

Hence, we obtained $dH_z(w) = w(x,y,z) - w(x,y,0)$

Taking derivative on both sides

$$0 = d^2 H_z(w) = dw(x,y,z) - d(w,y,0)$$
$$= -dw(x,y,0) \quad \because w \text{ closed}$$

$\Rightarrow w(x,y,0)$ is also closed.

Suffices to find an exact form for $w(x,y,0)$, for if we are able to do so i.e. $d\xi_1 = w(x,y,0)$, then $dH_z(w) = w - d\xi_1 \Rightarrow d(H_z w + \xi_1) = w$

$\Rightarrow w$ would be exact too.

Let $H_y w = \left(\int_0^y f(x,y,0)\, dy\right) dx + \left(\int_0^y h(x,y,0)\, dy\right) dz$

Then
$$dH_y w = \frac{d}{dz}\left(\int_0^y f(x,y,0)\, dy\right) dz \wedge dx + \frac{d}{dy}\left(\int_0^y f(x,y,0)\, dy\right) dy \wedge dx$$
$$+ \frac{d}{dx}\left(\int_0^y h(x,y,0)\, dy\right) dx \wedge dz + \frac{d}{dy}\left(\int_0^y h(x,y,0)\, dy\right) dy \wedge dz$$
$$= \left(\int_0^y (f_z - h_x)\, dy\right) dz \wedge dx + \left(f(x,y,0) - f(x,0,0)\right) dy \wedge dx$$
$$+ \left(h(x,y,0) - h(x,0,0)\right) dy \wedge dz$$
$$= \left(\int_0^y (-g_y)\, dy\right) dz \wedge dx + \left(f(x,y,0) - f(x,0,0)\right) dy \wedge dx$$
$$+ \left(h(x,y,0) - h(x,0,0)\right) dy \wedge dz$$
$$= \left(-g(x,y,0) + g(x,0,0)\right) dz \wedge dx + \left(f(x,0,0) - f(x,y,0)\right) dx \wedge dy$$
$$+ \left(h(x,0,0) - h(x,y,0)\right) dz \wedge dy$$
$$= w(x,0,0) - w(x,y,0)$$

Again, $0 = d^2 H_y w = dw(x,0,0) - d(w(x,y,0)) = dw(x,0,0) \quad \because w(x,y,0)$ closed

$\Rightarrow w(x,0,0)$ closed. $\therefore$ Suffices to find a closed form for $w(x,0,0)$

for if we can find ξ_2 s.t. $d\xi_2 = w(x,0,0)$, then

$$w(x,y,0) = w(x,0,0) - dH_y w = d\xi_2 - dH_y w$$
$$= d(\xi_2 - dH_y w)$$

$\Rightarrow \xi_1 = \xi_2 - dH_y w$. } next page.

Let $H_x(w) = \left(\int_0^x f(x,0,0)\,dx\right) dy - \left(\int_0^x g(x,0,0)\,dx\right) dz.$

Then, $dH_x(w) = \frac{d}{dx}\left(\int_0^x f(x,0,0)\,dx\right) dx \wedge dy + \frac{d}{dz}\left(\int_0^x f(x,0,0)\,dx\right) dz \wedge dy$

$- \frac{d}{dx}\left(\int_0^x g(x,0,0)\,dx\right) dx \wedge dz - \frac{d}{dy}\left(\int_0^x g(x,0,0)\,dx\right) dy \wedge dz$

$= \left(\int_0^x g_y + f_z\,dx\right) dz \wedge dy + \left(f(x,0,0) - f(0,0,0)\right) dx \wedge dy$

$+ \left(g(x,0,0) + g(0,0,0)\right) dx \wedge dz$

$= \int_0^x h_x\, dz \wedge dy + \left(f(x,0,0) - f(0,0,0)\right) dx \wedge dy$

$+ \left(g(0,0,0) - g(x,0,0)\right) dx \wedge dz$

$= \left(h(x,0,0) - h(0,0,0)\right) \cancel{dx} \, dz \wedge dy$

$+ \left(f(x,0,0) - f(0,0,0)\right) dx \wedge dy$

$+ \left(g(x,0,0) - g(0,0,0)\right) dz \wedge dx$

$= w(x,0,0) - w(0,0,0)$

Hence $\overset{0}{\shortparallel}\ d^2 H_x(w) = dw(x,0,0) - dw(0,0,0) = -dw(0,0,0) \Rightarrow w(0,0,0)$ is ~~exact~~ closed.

Hence, suffices to show $\exists\ \xi_3$ s.t. $d\xi_3 = w(0,0,0)$ for if that is the case, then $dH_x(w) = w(x,0,0) - d\xi_3 \Rightarrow w(x,0,0) = d(\xi_3 + H_x(w))$ is exact

Note $w(0,0,0)$ is a constant 2-form in $\mathbb{R}^3$ i.e. in the form of
$a\,dx \wedge dy + b\,dy \wedge dz + c\,dz \wedge dx.$

Note, if we let $\eta = ax\,dy + by\,dz + cz\,dx,$

then $d\eta = a\,dx \wedge dy + b\,dy \wedge dz + c\,dz \wedge dx = w(0,0,0)$

Hence $w(0,0,0)$ is exact $\Rightarrow w(x,0,0)$ is exact

$\Rightarrow w(x,y,0)$ is exact

$\Rightarrow w(x,y,z)$ is exact.

The explicit construction can be obtained via reverse engineering of all the homotopy operators.