MATH 105 HW1

7.2.1 The definition of outer product is $m^*(\Omega) := \inf\{\sum \mathrm{vol}(B_j) \mid (B_j)_j \text{ cover } \Omega\}$.

(v) Any box covers an empty set. $\forall \varepsilon$, can choose a box B with $\mathrm{vol}(B) < \varepsilon$ which still covers Ω. Hence, $m^*(\Omega) < \varepsilon \ \forall \varepsilon > 0 \Rightarrow \boxed{m^*(\Omega) = 0}$

(vi) Note $\sum \mathrm{vol}(B_j) \geq 0 \ \forall (B_j)_j$. Since $m^*(\Omega)$ is the infimum of a set of nonnegative values, $m^*(\Omega) \geq 0$ since 0 is always a lower bound.

(vii) Any sequence $(B_n)_n$ must be a cover of A if it is a cover of B.

Hence, $\{\sum \mathrm{vol}(B_j) \mid (B_j)_j \text{ covers } B\} \subseteq \{\sum \mathrm{vol}(B_j) \mid (B_j)_j \text{ covers } A\}$

In particular, the infimum of LHS $\geq$ infimum of RHS due to LHS being a subset of RHS

$\therefore m^*(B) \geq m^*(A) \Rightarrow \boxed{m^*(A) \leq m^*(B)}$ as desired.

(viii) Consider $n=2$. Suffices to show $m^*(A_1 \cup A_2) \leq m^*(A_1) + m^*(A_2)$

Since $m^*(A_1) = \inf\{\sum \mathrm{vol}(B_j) \mid (B_j)_j \text{ covers } A_1\}$, in particular, $m^*(A_1) + \frac{\varepsilon}{2}$ is not a lowerbound $\Rightarrow \exists$ a cover of A_1 with $\sum \mathrm{vol}(B_j) < m^*(A_1) + \frac{\varepsilon}{2}$.

Similarly, $\exists$ a cover of A_2 with $\sum \mathrm{vol}(B_j') < m^*(A_2) + \frac{\varepsilon}{2}$.

For any $x \in A_1 \cup A_2$, $x \in A_1$ or $x \in A_2 \Rightarrow x$ is covered by at least one open box in $(B_j)_j \cup (B_j')_j$

$$\Rightarrow m^*(A_1 \cup A_2) \leq \sum \mathrm{vol}(B_j) + \sum \mathrm{vol}(B_j')$$
$$\leq m^*(A_1) + \frac{\varepsilon}{2} + m^*(A_2) + \frac{\varepsilon}{2} = m^*(A_1) + m^*(A_2) + \varepsilon.$$

Since ε is arbitrarily chosen, $\boxed{m^*(A_1 \cup A_2) \leq m^*(A_1) + m^*(A_2)}$ as desired.

By induction on the number of sets. Suppose $m^*(A_1 \cup \dots \cup A_k) \leq \sum_{i=1}^{k} m^*(A_i)$

For $n = k+1$,

$$m^*((A_1 \cup \dots \cup A_k) \cup A_{k+1}) \leq m^*(A_1 \cup \dots \cup A_k) + m^*(A_{k+1})$$
$$\leq \sum_{i=1}^{k} m^*(A_i) + m^*(A_{k+1}) = \sum_{i=1}^{k+1} m^*(A_i)$$

$\therefore \boxed{m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m^*(A_i)}$ for $n \in \mathbb{N}$.

(x) Similar to (viii), note $m^*(A_i) + \frac{\varepsilon}{2^i}$ is not a lowerbound $\Rightarrow \exists$ a cover of A_i with $\sum \mathrm{vol}(B_j^{(i)}) < m^*(A_i) + \frac{\varepsilon}{2^i}$

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Hence, $\forall x \in \bigcup_{j\in J} A_j$, $x \in \bigcup_{i,j} B_j^{(i)}$ $\Rightarrow$ x is covered by at least one box in the sequence of sequence of boxes

$\Rightarrow m^*\left(\bigcup_{j\in J} A_j\right) \leq \sum_{i=1}^{\infty}\left[m^*(A_i) + \frac{\varepsilon}{2^i}\right] = \sum_{i=1}^{\infty} m^*(A_i) + \varepsilon$

Since ε is arbitrary, $\boxed{m^*\left(\bigcup_{j\in J} A_j\right) \leq \sum_{i=1}^{\infty} m^*(A_i)}$

(xiii) Let $\{B_j\}_j$ be a sequence of open boxes that cover Ω.
Then consider $\{x+B_j\}_j$ where $x+B_j$ denote box B_j translated by x (still any open box with axes parallel). This is an open cover for $x+\Omega$, since

$y \in x+\Omega \Rightarrow y-x \in \Omega \Rightarrow y-x$ covered by B_k for some k
$\Rightarrow y$ covered by $x+B_k$ for the same k.

Similarly, any open cover of $x+\Omega$ is an open cover of Ω.
Hence, $m^*(x+\Omega) \leq m^*(\Omega)$ and $m^*(\Omega) \leq m^*(x+\Omega)$

$\Rightarrow \boxed{m^*(x+\Omega) = m^*(\Omega)}$ as desired.

7.22 Let $(B_i)_i$ be a sequence of open boxes that cover A with total volume $a < \infty$ and $(B_i')_i$ be a sequence of open boxes that cover B. with total volume $b < \infty$

Then, the set $\{B_i \times B_j' \mid i \in \mathbb{N}, j \in \mathbb{N}\}$ is countable (Same reason why $\mathbb{Q}$ is countable). Each of $B_i \times B_j'$ is an open rectangle and the total volume is ab by limit theorem of sequence of partial sums. ~~$\Rightarrow$ total volume~~

$\forall x \in A \times B$, $x \in \bigcup_{i,j} B_i \times B_j'$ $\Rightarrow m^*_{n+m}(A \times B) \leq ab$.

Since $m_n^*(A) + \varepsilon$ is not a lower bound ~~for upper~~ for outer measure of a cover on A and $m_m^*(B) + \varepsilon$ is not a lower bound for outer measure of a cover on B, $\exists$ covers $(B_i)_i$ and $(B_j')_j$ s.t. $\sum \mathrm{vol}(B_i) < m_n^*(A) + \varepsilon$ and $\sum \mathrm{vol}(B_j') < m_m^*(B) + \varepsilon$

$\Rightarrow m^*(A \times B) \leq (m^*(A) + \varepsilon)(m^*(B) + \varepsilon) = m^*(A)\, m^*(B) + \varepsilon(m^*(A) + m^*(B)) + \varepsilon^2$

Taking $\varepsilon \to 0$,

$\boxed{m^*(A \times B) \leq m^*(A)\, m^*(B)}$ as desired.

7.2.3 (a) $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$

$\forall n \in \mathbb{N}$, $A_n \subset \bigcup_{j=1}^{n} A_j \Rightarrow m(A_n) \leq m(\bigcup_{j=1}^{\infty} A_j) \; \forall n \in \mathbb{N}$

Consider the value sequence $(m(A_n))_n$. Since $m(\bigcup_{j=1}^{\infty} A_j)$ is bigger than every element, $m(\bigcup_{j=1}^{\infty} A_j) \geq \lim_{n\to\infty} m(A_n)$

Now consider the value sequence $\left(m(\bigcup_{j=1}^{n} A_j)\right)_n$. Note that $(m(A_n))_n$ is an increasing sequence. Hence $\lim_{j\to\infty} m(A_j) \geq m(\bigcup_{j=1}^{n} A_j) = m(A_n) \; \forall n.$

$\Rightarrow \lim_{j\to\infty} m(A_j) \geq m(\bigcup_{j=1}^{\infty} A_j)$ since LHS is bigger than $\forall n.$

$\therefore \boxed{\lim_{j\to\infty} m(A_j) = m(\bigcup_{j=1}^{\infty} A_j)}$

(b) $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \quad m(A_1) < \infty$

Note since $\bigcap_{j=1}^{\infty} A_j \subseteq A_n \; \forall n \in \mathbb{N}$, by monotonicity, we have $m(\bigcap_{j=1}^{\infty} A_j) \leq m(A_n) \; \forall n$

$\Rightarrow m(\bigcap_{j=1}^{\infty} A_j) \leq \lim_{j\to\infty} m(A_j)$, since LHS is less than every value on RHS sequence.

Now consider the sequence $\left(m(\bigcap_{j=1}^{n} A_j)\right)_n = (m(A_n))_n$. This sequence is decreasing (due to monotonicity).

Hence, $\forall n \in \mathbb{N}$, $m(\bigcap_{j=1}^{n} A_j)$ is a lim sup of the sequence $(m(A_n))_n$

$\Rightarrow m(\bigcap_{j=1}^{n} A_j) \geq \lim_{j\to\infty} m(A_j) \; \forall n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} m(\bigcap_{j=1}^{n} A_j) \geq \lim_{j\to\infty} m(A_j)$

as desired. $\boxed{\lim_{j\to\infty} m(A_j) = m(\bigcap_{j=1}^{\infty} A_j)}$

7.2.4. $q \in \mathbb{Z}$. Consider the q^n translates of $[0, \frac{1}{q})^n$ by the vector $x = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ where $a_i \in \{0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}\}$

The q^n translates are mutually disjoint and a subset of $[0,1]^n$. By translational invariance and normalization and monotonicity,

$q^n m([0, \frac{1}{q})^n) \leq 1 \Rightarrow \boxed{m([0, \frac{1}{q})^n) \leq q^{-n}}$ as desired

7.2.4 continued | Similarly, by considering the same translation to $[0,\frac{1}{q}]^n$ instead, by finite subadditivity and translational invariance,

$$q^n m([0,\tfrac{1}{q}]^n) \ge \underbrace{m([0,1])}_{\text{Since it is the union of the } q^n \text{ boxes}} = 1 \Rightarrow m([0,\tfrac{1}{q}]^n) \ge \frac{1}{q^n} = q^{-n}$$

Now consider $m([0,\frac{1}{q}]^n \setminus (0,\frac{1}{q})^n)$

A n-dimensional cube has $n\cdot 2^{n-1}$ edges. We can cover each edge with a box of volume $\frac{\varepsilon}{n\cdot 2^{n-1}}$ with one side $> \frac{1}{q}$. Hence, $m([0,\frac{1}{q}]^n \setminus (0,\frac{1}{q})^n) < \frac{\varepsilon}{n\cdot 2^{n-1}}\cdot(n\cdot 2^{n-1}) = \varepsilon$

By ~~countable~~ finite additivity, $m([0,\frac{1}{q}]^n) = m((0,\frac{1}{q})^n) + m([0,\frac{1}{q}]^n \setminus (0,\frac{1}{q})^n)$

$$\Rightarrow m([0,\tfrac{1}{q}]^n) - m((0,\tfrac{1}{q})^n) = m([0,\tfrac{1}{q}]^n \setminus (0,\tfrac{1}{q})^n) < \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow m([0,\tfrac{1}{q}]^n) - m((0,\tfrac{1}{q})^n) = 0$$

and the only way for this to hold is when

$$m([0,\tfrac{1}{q}]^n) = m((0,\tfrac{1}{q})^n) = \frac{1}{q^n} = \boxed{q^{-n}}$$

7.4.1 Let $A = (a,b)$. If $A \subset (0,\infty)$ or $A \subset [-\infty, 0]$, then we are done, since RHS $= m^*(A) + m^*(\emptyset) = m^*(A)$. Suppose otherwise. Then $b > 0$ and $a \le 0$.

$$\Rightarrow A \cap (0,\infty) = (0,b), \quad A \setminus (0,\infty) = (a,0]$$

Since for any box, outer measure returns the volume of it,

$$\text{RHS} = m^*(A \cap (0,\infty)) + m^*(A \setminus (0,\infty)) = m^*((0,b)) + m^*((a,0])$$
$$= b + (-a) = b - a = m^*((a,b)) = \text{LHS}$$

Hence if A is an open interval in $\mathbb{R}$, then $\boxed{m^*(A) = m^*(A \cap (0,\infty)) + m^*(A \setminus (0,\infty))}$

7.4.2 A is an open box in $\mathbb{R}^n$. If $A \cap E = A$ or $\emptyset$, then we are done since RHS is either $m^*(A) + m^*(\emptyset)$ or $m^*(\emptyset) + m^*(A)$, either way $= m^*(A) =$ LHS.

* Else, by 7.4.1, we can ~~state~~ consider two intervals $x_n > 0$ and $x_n \le 0$ for the last dimension. This gives us two boxes (one ~~open~~ open, one half open). The remaining dimensions are unaffected by the half plane and hence,

$$m^*(A \cap E) + m^*(A \setminus E) = \underbrace{\text{vol}_{n-1}}_{n-1 \text{ dimensional volume}} \cdot m^*((a,0]) + \text{vol}_{n-1}\, m^*((0,b)) = \text{vol}_{n-1}(b-a)$$
$$= m^*(A) \text{ as desired}$$

7.4.3. E is measurable if $\forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$
let $(B_i)_i$ be a sequence of boxes that cover A. Then, the sequence of boxes $(B_i \cap E)_i$ "almost" covers $A \cap E$ and the sequence of boxes $(B_i \setminus E)_i$ "almost" covers $A \setminus E$

The "almost" is because of the half plane, but we can always cover it with a sequence of open boxes with ~~...~~ outer measure 0.

Since $m^*(B_i) = m^*(B_i \cap E) + m^*(B_i \setminus E)$, any upper bound of $m^*(A)$ is automatically an upper bound for $m^*(A \cap E) + m^*(A \setminus E)$. — ①

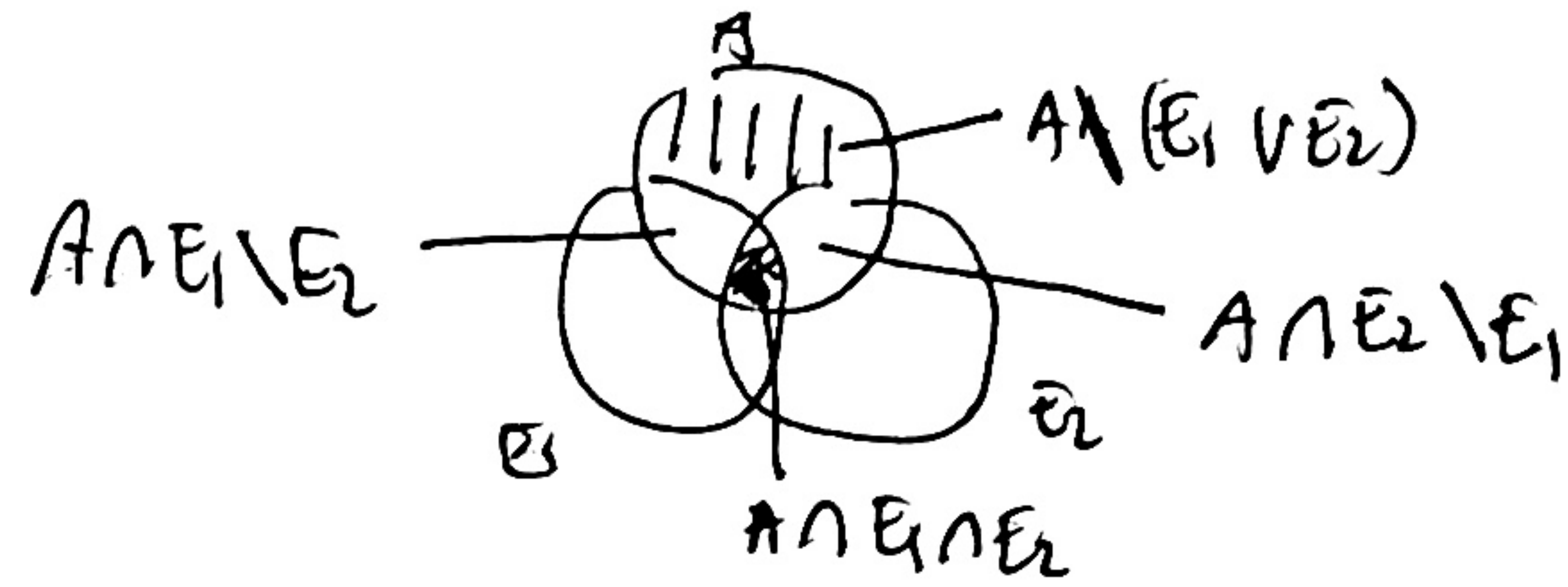
Conversely, if $(C_i)_i$ and $(D_i)_i$ are sequences of boxes that cover $A \cap E$ and $A \setminus E$, then $(C_i)_i \cup (D_i)_i$ covers A. In particular any upper bound of ~~$m^*(A \cap E)$~~ $m^*(A \cap E) + m^*(A \setminus E)$ is automatically an upper bound for $m^*(A)$. — ②

①, ② $\Rightarrow$ $\boxed{m^*(A) = m^*(A \cap E) + m^*(A \setminus E)}$ as desired.

7.4.4. (a) E is measurable $\Rightarrow \forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$.
Note that, $m^*(A \cap E) = m^*(A \setminus (\mathbb{R}^n \setminus E))$
and $m^*(A \setminus E) = m^*(A \cap (\mathbb{R}^n \setminus E))$
$\Rightarrow \forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E) = m^*(A \setminus (\mathbb{R}^n \setminus E)) + m^*(A \cap (\mathbb{R}^n \setminus E))$

Hence, by definition, $\boxed{\mathbb{R}^n \setminus E \text{ is also measurable.}}$ *

(b) E is measurable $\Rightarrow \forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$
Now, consider $x + E$ where $x \in \mathbb{R}^n$.
$\forall A \subset \mathbb{R}^n$
let $Y = A - x$. Then $m^*(Y) = m^*(Y \cap E) + m^*(Y \setminus E)$ since E measurable
$\Rightarrow m^*(A - x) = m^*((A - x) \cap E) + m^*((A - x) \setminus E)$
$\Rightarrow m^*(A) = m^*(A \cap (E + x)) + m^*(A \setminus (E + x))$
(due to translational invariance of outer measure)
$\Rightarrow \boxed{m(x + E) \text{ is (translation invariant) measurable}}$



7.4.4 (c) E_1, E_2 measurable $\Rightarrow \forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A\cap E_1) + m^*(A\setminus E_1)$
$m^*(A) = m^*(A\cap E_2) + m^*(A\setminus E_2)$

Following the hint, I will show:

$$m^*(A) = m^*(A\cap E_1\cap E_2) + m^*(A\cap E_1\setminus E_2) + m^*(A\cap E_2\setminus E_1) + m^*(A\setminus(E_1\cup E_2))$$

Consider $A\leftarrow A\cap E_1$ and $A\leftarrow A\setminus E_1$ yields:

$m^*(A\cap E_1) = m^*(A\cap E_1\cap E_2) + m^*(A\cap E_1\setminus E_2)$ (since E_2 measurable)

$m^*(A\setminus E_1) = m^*(A\setminus E_1\cap E_2) + m^*(A\setminus E_1\setminus E_2)$

$= m^*(A\cap E_2\setminus E_1) + m^*(A\setminus(E_1\cup E_2))$

$\therefore m^*(A) \underset{E_1 \text{ measurable}}{=} m^*(A\cap E_1) + m^*(A\setminus E_1) = m^*(A\cap E_1\cap E_2) + m^*(A\cap E_1\setminus E_2)$
$+ m^*(A\cap E_2\setminus E_1) + m^*(A\setminus(E_1\cup E_2))$

$\therefore \boxed{m^*(A) = m^*(A\cap E_1\cap E_2) + m^*(A\cap E_1\setminus E_2) + m^*(A\cap E_2\setminus E_1) + m^*(A\setminus(E_1\cup E_2))}$

As desired.

It is clear that $m^*(A\cap(E_1\cap E_2)) + m^*(A\setminus(E_1\cap E_2))$
$\underset{\text{finite sub-additivity}}{\geq} m^*((A\cap(E_1\cap E_2))\cup(A\setminus(E_1\cap E_2))) = m^*(A)$

On the other hand,

$m^*(A) = m^*(A\cap E_1\cap E_2) + m^*(A\cap E_1\setminus E_2) + m^*(A\cap E_2\setminus E_1) + m^*(A\setminus(E_1\cup E_2))$
$\underset{\text{finite sub-additive}}{\geq} m^*(A\cap E_1\cap E_2) + m^*(A\setminus(E_1\cap E_2))$

$\therefore \boxed{m^*(A) = m^*(A\cap E_1\cap E_2) + m^*(A\setminus(E_1\cap E_2))}$ $\forall A$

$\therefore \boxed{E_1\cap E_2 \text{ is measurable.}}$

Similarly,

$m^*(A\cap(E_1\cup E_2)) + m^*(A\setminus(E_1\cup E_2)) \underset{\text{finite } \cancel{\text{countable}} \text{ sub-additive}}{\geq} m^*(A)$

On the other hand,

$m^*(A) = m^*(A\cap E_1\cap E_2) + m^*(A\cap E_1\setminus E_2) + m^*(A\cap E_2\setminus E_1) + m^*(A\setminus(E_1\cup E_2))$
$\underset{\text{finite subadditive}}{\geq} m^*(A\cap(E_1\cup E_2)) + m^*(A\setminus(E_1\cup E_2))$

$\therefore \boxed{m^*(A) = m^*(A\cap(E_1\cup E_2)) + m^*(A\setminus(E_1\cup E_2))}$ $\forall A$

$\therefore \boxed{E_1\cup E_2 \text{ is measurable}}$

7.4.4 (d) We proceed with induction on N. By (c), $N=2$ is true.
Suppose $N=k$ is true. Then for $N=k+1$, measurable by inductive hypothesis
using induction hypothesis for $N=2$, measurable

$$E_1 \cup E_2 \cup \dots \cup E_k \cup E_{k+1} = (E_1 \cup E_2 \cup \dots \cup E_k) \cup E_{k+1}$$

is measurable.

Similarly, $E_1 \cap E_2 \cap \dots \cap E_k \cap E_{k+1} = (E_1 \cap E_2 \cap \dots \cap E_k) \cap E_{k+1}$ is measurable.

$\therefore$ $\boxed{\bigcap_{j=1}^{N} E_j \text{ and } \bigcup_{j=1}^{N} E_j \text{ is measurable } \forall N \in \mathbb{N}}$

(e) We know that by Lemma 7.4.2 that every half-space $\{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n > 0\}$ is measurable.
We can treat each open box as the intersection of $2n$ half-spaces
(Suppose the open box is defined as $(x_1', y_1') \times (x_2', y_2') \times \dots \times (x_n', y_n')$), then the half planes are $\{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_i > x_i'\}$ and $\{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_i < y_i'\}$
Hence, by Lemma 7.4.4 (d), the intersection is measurable
$\Rightarrow$ $\boxed{\text{the open box is measurable}}$

Since half planes of the form $\{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n > 0\}$ are measurable, then $\mathbb{R}^n \setminus \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n > 0\} = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \leq 0\}$ is measurable. Hence again, we can treat closed box as the intersection of $2n$ of such "closed" half planes $\Rightarrow$ $\boxed{\text{the closed box is measurable}}$

(f) Suffices to show $\forall A$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$
Since $A \cap E \subset E$, $m^*(A \cap E) \leq m^*(E) = 0$
$\Rightarrow m^*(A \cap E) = 0 \ \forall A$.

finite subadditivity

Note, we have $m^*(A \cap E) + m^*(A \setminus E) \geq m^*(A)$
and $m^*(A) \geq m^*(A \setminus E)$ (monotonicity)

$\therefore$ $m^*(A) = m^*(A \setminus E) \ \forall A$
$\therefore$ $\boxed{\text{Any set } E \text{ with outer measure } 0 \text{ is measurable.}}$