MATH 105 HW 1 (After Feedback)

7.2.3 $m\left(\bigcup_{j=1}^{\infty} A_j\right) = m(A_n) + \sum_{j=n}^{\infty} m(A_{j+1} \setminus A_j)$ (countable additivity of disjoint sets)

$$= m(A_n) + \sum_{j=n}^{\infty} \left[m(A_{j+1}) - m(A_j)\right]$$

Taking limit as $n \to \infty$ on both sides, LHS remains unchanged since it does not depend on n. so

$$m\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty}\left(m(A_n) + \sum_{j=n}^{\infty}\left[m(A_{j+1}) - m(A_j)\right]\right)$$

Note $(m(A_j))_j$ is an increasing sequence, so the sets get larger and $A_j \subset A_{j+1}$. Hence, assuming $\lim_{j\to\infty} m(A_j)$ exists, since

$$m(A_n) + \sum_{j=n}^{\infty}\left[m(A_{j+1}) - m(A_j)\right] = \lim_{j\to\infty} m(A_j),$$

$$\lim_{n\to\infty} \sum_{j=n}^{\infty}\left[m(A_{j+1}) - m(A_j)\right] + \lim_{n\to\infty} m(A_n) = \lim_{j\to\infty} m(A_j) \Rightarrow \lim_{n\to\infty} \sum_{j=n}^{\infty}\left[m(A_{j+1}) - m(A_j)\right] = 0$$

Hence $m\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty}\left(m(A_n)\right) + \lim_{n\to\infty}\sum_{j=n}^{\infty}\left[m(A_{j+1}) - m(A_j)\right] = \lim_{n\to\infty} m(A_n)$

$\therefore \boxed{m\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} m(A_n)}$ as desired.

(b). Consider the sequence $B_j = A_1 \setminus A_j$. Then B_j is an increasing sequence of measurable sets with $B_j \subseteq B_{j+1}$. Following the result of (a),

$$m\left(\bigcup_{j=1}^{\infty} B_j\right) = \lim_{n\to\infty} m(B_n)$$

Note $A_1 \setminus \bigcup_{j=1}^{\infty} B_j = \bigcap_{j=1}^{\infty} A_j \Rightarrow m\left(\bigcap_{j=1}^{\infty} A_j\right) = m(A_1) - m\left(\bigcup_{j=1}^{\infty} B_j\right)$

$$= m(A_1) - \lim_{n\to\infty} m(B_n)$$

$$= \lim_{n\to\infty}\left(m(A_1) - m(B_n)\right) = \lim_{n\to\infty} m(A_1 \setminus B_n)$$

$$= \lim_{n\to\infty} A_n$$

$\therefore \boxed{m\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} A_n}$