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MATH 105 HW 3

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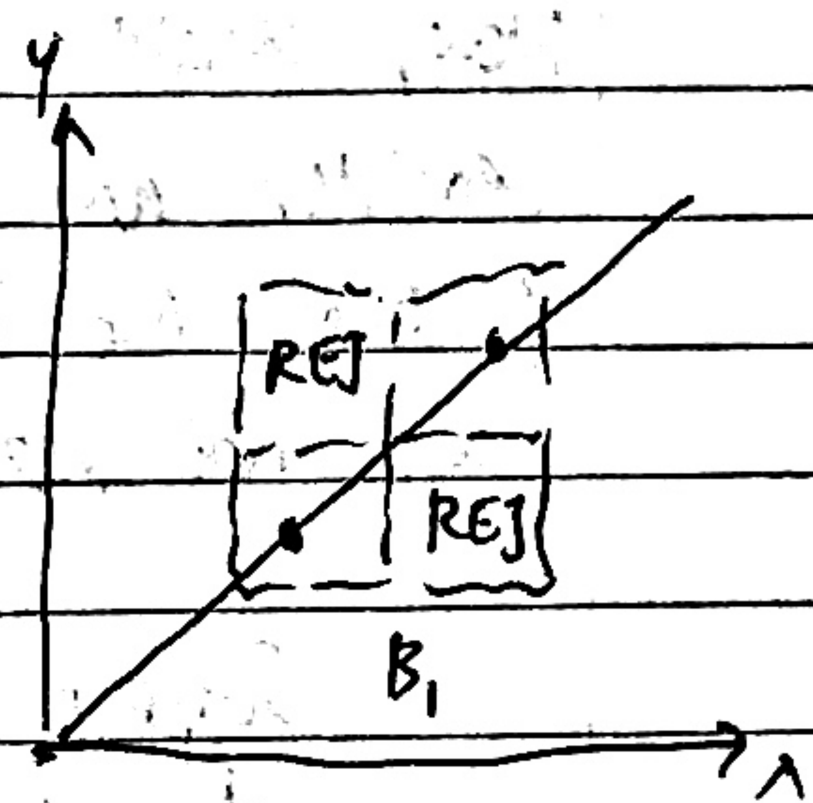


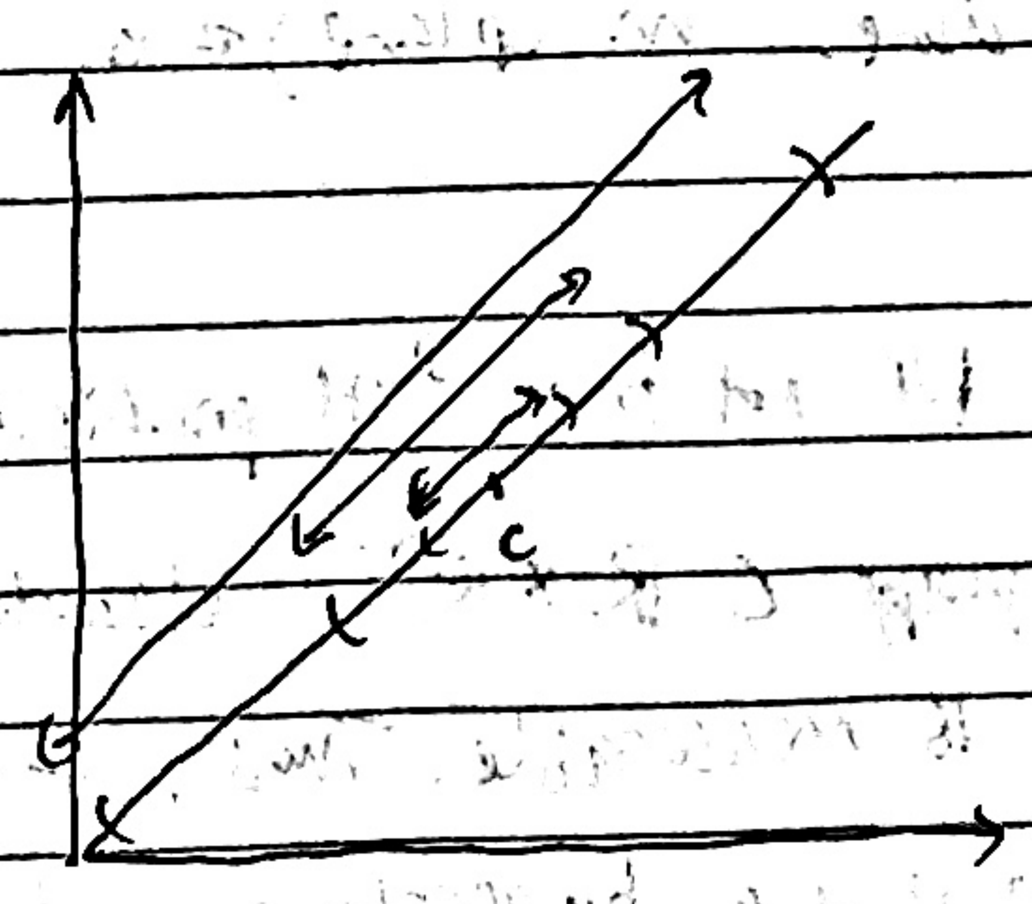
[Pugh 3] 1. Firstly, I will prove the special case $\{y = x\}$. Consider a line segment first. Consider a box B_1 open that covers this line segment.

Now, we divide B_1 into four boxes, reject the top-left and bottom right ones, and extend the remaining two boxes slightly so that they cover any middle points, as illustrated in the diagram. We can do this such that the total volume of the covered boxes is $\leq \frac{2}{3}|B_1|$.

Note that we can continue repeating this algorithm and this will still give us a finite cover of the line segment with total volume $\leq \left(\frac{2}{3}\right)^n |B_1|$ after n iterations.

$\therefore m^*(\text{line segment}) \leq \left(\frac{2}{3}\right)^n |B_1| \quad \forall n \Rightarrow \boxed{m^*(\text{line segment}) = 0}$

For an infinitely long segment, we can treat it as follows: we pick a center c



Consider the line segments $[c - 2^n, c + 2^n]$ with 2^n being the length of the segment.

For each $n \in \mathbb{N}$, we can cover the line with boxes of total volume $< \frac{\varepsilon}{2^n}$

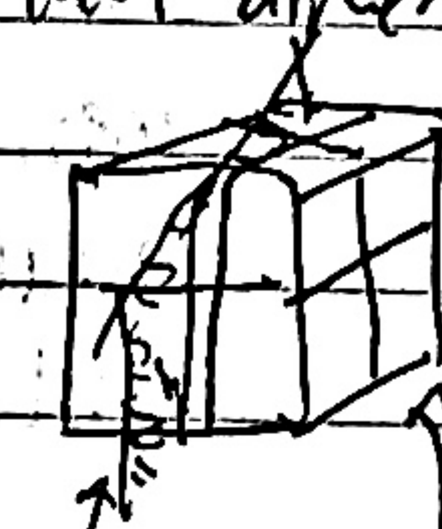
Hence, the entire line segment can be covered by boxes of volume $< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \dots = \varepsilon$

Since ε is arbitrarily chosen, the entire diagonal line has outer measure 0 and thus is a zero set.

next page for generalization.

Now, consider a general n-dimension plane ~~of~~ that belongs in dimension $< n$.
Consider an open box B that has a nonempty intersection with this plane. Partition the box into 2^n smaller boxes (2 for each dimension). The plane does not intersect all 2^n of these smaller boxes.

Suppose otherwise that the plane intersects all of these 2^n boxes, pick a dimension. The plane intersects all 2^{n-1} boxes in the $n-1$ other dimension
$\Rightarrow$ the plane must be restricted to ~~that~~ a range in that dimension, meaning it will not intersect the other 2^{n-1} boxes.



Hence, the total volume of required boxes drop by at least a factor of $\frac{2^n - 1 + \frac{1}{2}}{2^n}$ for safety

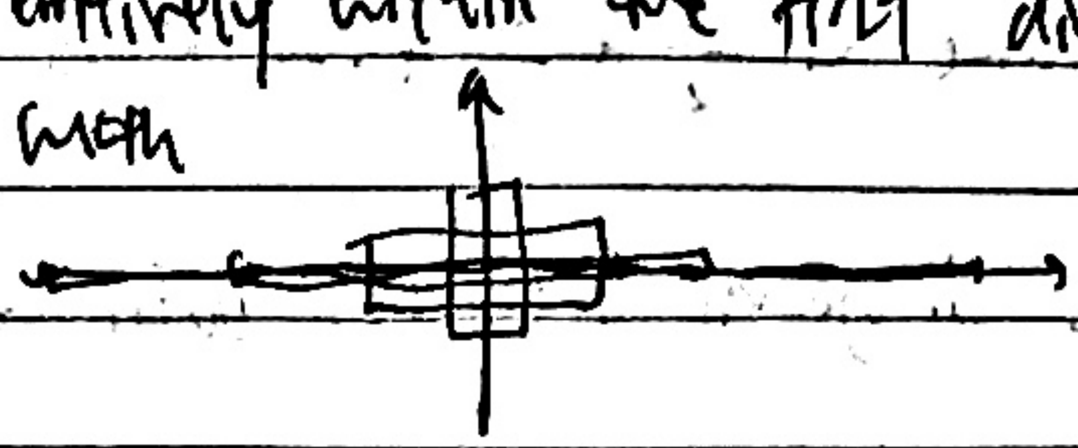
$$= \frac{2^n - \frac{1}{2}}{2^n} < 1$$

After many iterations of this algorithm, we still produce a covering of the plane, and $m^*(\text{plane}) < \left(\frac{2^n - \frac{1}{2}}{2^n}\right)^k$ where k is the number of iterations.

Since $\lim_{k \to \infty} \left(\frac{2^n - \frac{1}{2}}{2^n}\right)^k = 0$, we are done. $m^*(\text{plane}) = 0$.

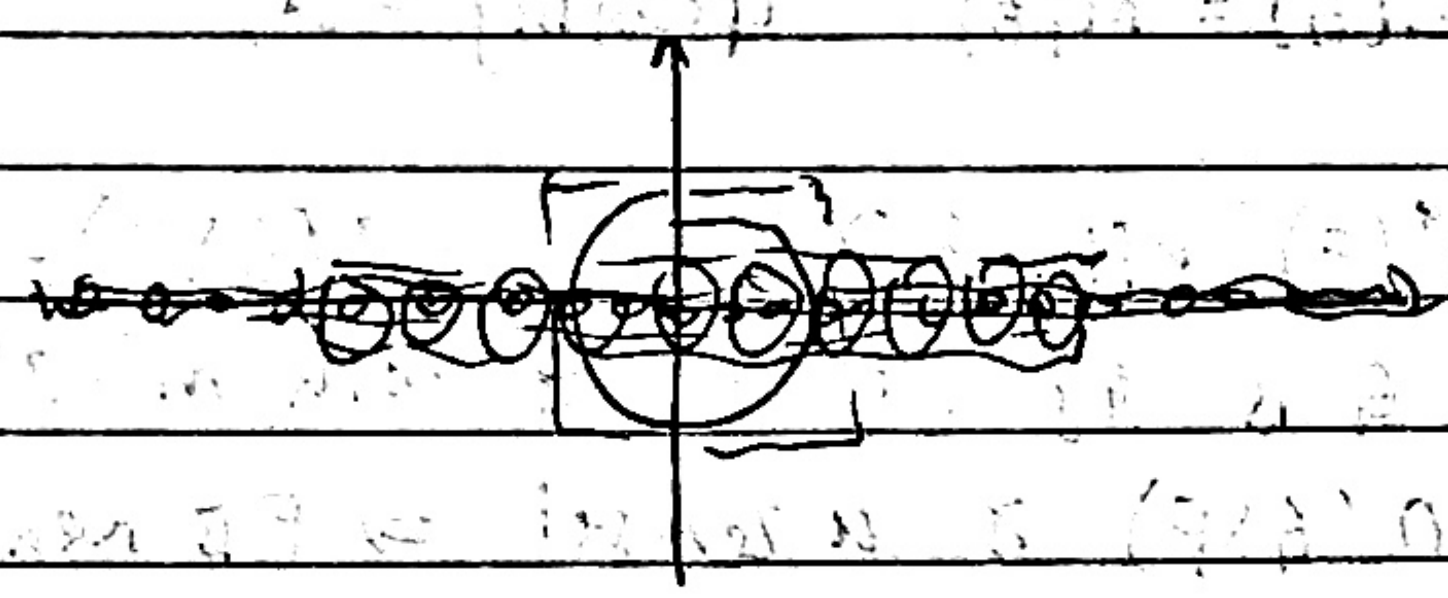
Two other ways which I believe will work, but not from first principles.

① By zero slice theorem, let E denote the plane $\subset \mathbb{R} \times \mathbb{R}^{n-1}$. Clearly, a plane is closed (the complement is open) and hence is measurable. Thus, if the plane lies entirely within the first dimension $\mathbb{R}$, then it is a hyperplane and we can solve it with



Else, every slice of E is a zero set, since every slice of E wrt the plane ~~at~~ ^{m} ~~a~~ $n-1$ dimensional plane which has measure 0 by induction.
$\Rightarrow$ $\boxed{E = \text{plane is a measure } 0 \text{ set}}$

② Balls are open and can therefore be treated as countable union of open cubes and a measure 0 set. More balls are invariant to rotations etc, and we can thus transform our arguments for hyperplanes having measure 0 by using balls



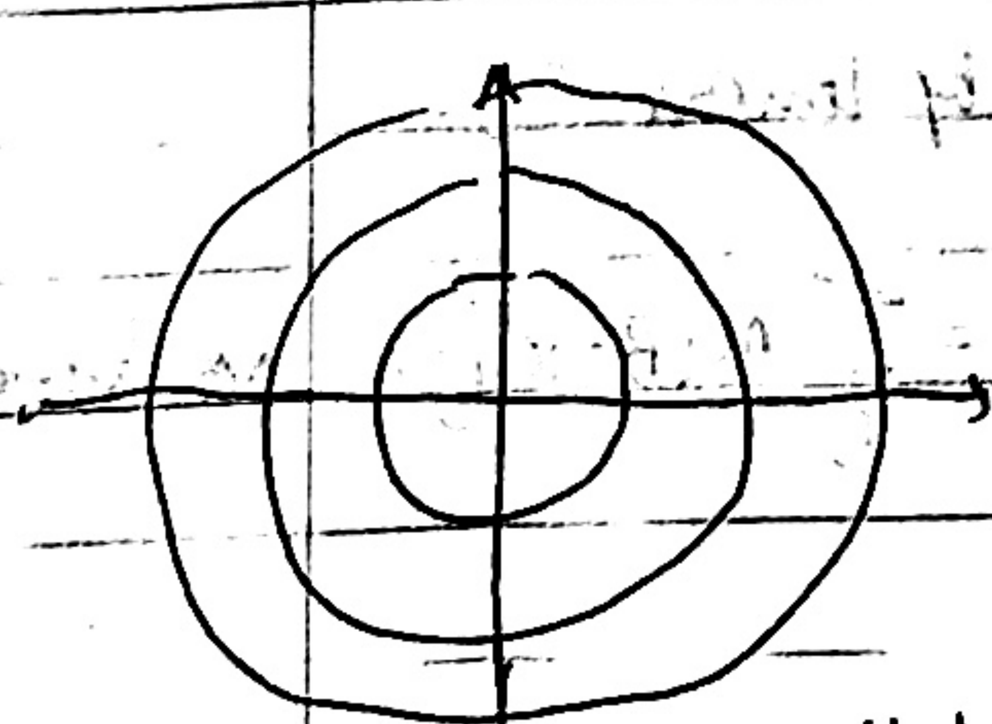
Hence for arbitrary direction, we can just consider the translation of these balls (equivalently the open boxes that make up these balls)
$\therefore$ Measure of affine plane $= 0$.

[Pugh 6] 2. 11. Theorem: Each measurable set E can be sandwiched between an F_σ set and a G_δ set, $F \subset E \subset G$ s.t. $G \setminus F$ is a zero set. Conversely, if $\exists F \subset E \subset G$, then E is measurable.

Firstly, we proceed with bounded $E \subset \mathbb{R}^n$. Let R be a box that contains E. By measurability of E, $m^*R = m^*E + m^*(R \setminus E)$. Similar to proof in textbook, $\exists$ $(U_n)_n$ and $(V_n)_n$ open s.t. $\lim_{n\to\infty} m^*(U_n) = m^*(E)$, $\lim m^*(V_n) = m^*(E^c)$

Since E measurable, $m^*(U_n) = m^*(U_n \cap E) + m^*(U_n \setminus E) = m^*(E) + m^*(U_n \setminus E)$
$\Rightarrow m^*(U_n \setminus E) \to 0$, $m^*(V_n \setminus (R \setminus E)) \to 0$. Denote $K_n = R \setminus V_n$, then K_n forms an increasing sequence of closed set s.t. $m^*(K_n) \to m^*(E)$. Hence, denote $F = \bigcup K_n$ and $G = \bigcap U_n$ $\Rightarrow$ $m^*(F) = m^*(E) = m^*(G)$. Consequently, $m(G \setminus F) = 0$.

For unbounded case, consider splitting $\mathbb{R}^n$ into countable disjoint regions by balls

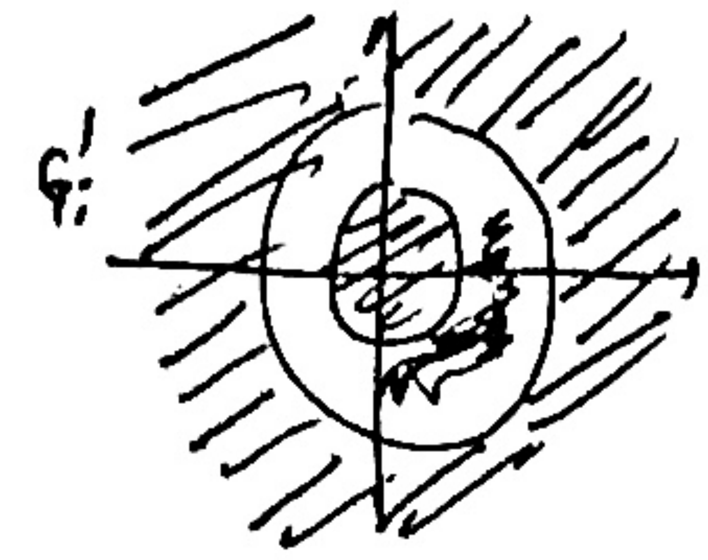


Let $E_i := E \cap (B_0(i) - B_0(i-1))$ intersection with ball i minus ball $i-1$.
$i \in \mathbb{N}$

E is measurable $\Rightarrow$ intersecting with also measurable.
$\Rightarrow E_i$ are all measurable.

For each E_i, consider $F_i = \bigcup K_n^{(i)}$. Consider the union of all such F_σ set across all i. Since union of F_σ set is still a F_σ set, we have a F_σ set F s.t. $m^*F = m^*E$ (possibly $= \infty$)

For each G_δ set, i.e. for each i, consider the $G_i = \bigcap U_n^{(i)}$ the G_δ set covering E_i.
Consider $G_i' = \bigcap (U_n^{(i)} \cup B_0(i-1) \cup \overline{B_0(i)}^c)$ i.e. intersect with open ball of radius $i-1$, and complement of closed ball i.



$G_i' = \bigcap (U_n^{(i)} \cup B_0(i-1) \cup \bar{B}_0(i)^c)$

Note $U_n^{(i)} \cup B_0(i-1) \cup \bar{B}_0(i)^c$ is an open set and therefore G_i' is a G_δ set.
Consider $\bigcap G_i'$, this covers all of E, and is a G_δ set. Hence, we have a G_δ set $G' = \bigcap_i G_i'$ s.t. $m^*(G) = m^*(E)$ (possibly $= \infty$).

Since $M^*(G) = M^*(E) = M^*(F)$ with $G \supset E \supset F$, $M^*(G \setminus F) = 0$.
Conversely, if F is F_σ, G is G_δ, $F \subset E \subset G$ with $m^*(G \setminus F) = 0$, then $E = F \cup Z$ where $Z = E \cap (G \setminus F)$ is a zero set $\Rightarrow$ E is measurable

21: Measurable Product Theorem: If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ are measurable, then $A \times B$ is measurable and $m(A \times B) = m(A) \cdot m(B)$.

Since zero sets are proved, assume $m(A), m(B) \neq 0$. Suffices to show lemma 23, lemma 24 in higher dimensions.

For lemma 23, suppose $m(A) = 0$. Then, we can cover A with open intervals I_i with length $< \frac{\varepsilon}{2^i (2l)^k}$ where k is dimension of B. Then the volume of $I_i \times [-l, l]^k$ is less than $\frac{\varepsilon}{2^i}$. The union of all these rectangles cover $A \times \mathbb{R}^k$ and has measure $< \varepsilon$ $\forall \varepsilon$ $\Rightarrow$ $A \times B$ is zero set when A is zero set.

For lemma 24, $U \times V$ is open $\Rightarrow$ measurable. By lemma 16, $U = \bigsqcup_i B_i \cup Z_u$ and $V = \bigsqcup_j C_j \cup Z_v$ where B_i, C_j are boxes and Z_u, Z_v are zero sets.

$U \times V = \bigsqcup_{ij} B_i \times C_j \cup Z$, a measure 0 set by lemma 23.

Since $\sum_i M(B_i) \sum_j M(C_j) = \sum_{ij} M(B_i) M(C_j) = \sum_{ij} M(B_i \times C_j)$, we have:
$\boxed{M(U \times V) = MU \cdot MV.}$

For the actual theorem, the claim that the hull/kernel of a product is the product of hull/kernel still holds since the proofs did not use lower dimension arguments

next page.

We can proceed by partitioning the space into $[0,1]^n$ cubes for A and $[0,1]^k$ cubes for B. We apply the same arguments: U_n, V_n be sequences of open sets in I^n, I^k converging down to H_A, H_B, then $U_n \times V_n$ is a sequence of open sets in I^{n+k} converging down to $H_A \times H_B$.

Applying downward measure continuity, $m(U_n \times V_n) \to m(H_A \times H_B)$. By Lemma 24, we have $m(U_n \times V_n) = m(U_n) \cdot m(V_n)$ which $\to m(A), m(B)$ respectively. Hence, $\boxed{m(A \times B) = m(A) \cdot m(B)}$

Pugh 11 / 12 — 3. Firstly, I will prove Pugh 11 (c). Note $m^*(A) \le J^*(A)$ because $J^*(A)$ requires a finite covering of A with open boxes, while $m^*(A)$ just requires countable. Due to the increased restriction, $m^*(A) \le J^*(A)$ (taking infimum over a smaller set).

If A is compact, by Heine Borel's theorem, any open cover of A has a finite subcover. Hence, for any countable open cover of A, we can achieve the same, if not better for $J^*(A)$. Hence $m^*(A) \ge J^*(A)$. Combined with the above inequality $\boxed{m^*(A) = J^*(A) \text{ if } A \text{ compact}}$

Now, moving onto 12. Clearly from 11(c), since A is assumed to be a bounded set $\bar{A}$ is compact $\Rightarrow \boxed{J^*\bar{A} = m\bar{A}}$

Clearly, we also have $J^*A \le J^*\bar{A}$ by monotonicity of J.
Suffices to show $J^*A + \varepsilon \ge J^*\bar{A}$ $\forall \varepsilon$.
Let $B_1, \dots, B_N$ be the open intervals s.t. $\cup B_i$ covers A and $\sum |B_i| < J^*A + \frac{\varepsilon}{2}$.
Enlarge each B_i in all dimensions s.t. their volume becomes $|B_i| + \frac{\varepsilon}{2 \cdot 2^i}$
i.e. all dimensions of B_i increase by at least a positive constant $\delta_i > 0$.
I claim that this new sequence $B_1', B_2', \dots B_N'$ covers $\bar{A}$.

Suppose otherwise, then $\exists x \in \bar{A}$ that is not covered. ~~This means $\forall r > 0$~~, since $x \in \bar{A}$, ~~it means $\forall r >$~~ x definitely cannot $\in A$ since the boxes $\cup B_i$ already cover A and B_i' are an enlargement of the boxes

next page

Hence $x \in \bar{A} \setminus A \Rightarrow \forall \overset{\in \mathbb{Z}}{n} > 0, \exists x_n$ s.t. $x_n \in B_x(\frac{1}{n})$ that is covered (since x is on the boundary and has non empty intersection with A).

~~Consider this sequence (x_n). Since $\bar{A}$ compact, $\exists$ subsequence $(x_{n_k})_k$ that converges~~

In particular, $\exists x_i$ s.t. $\frac{1}{i} < \dfrac{\overbrace{\min(\delta_1, \delta_2, \ldots, \delta_i)}^{\text{enlargements of the boxes}}}{\sqrt{n}}$ ← dimension of the space

s.t. x_i is covered in a box, say B_i

Thus, the enlargement of box would have covered x, as desired. (contradiction!) since we assumed x is not covered.

Hence $J^* A + \varepsilon \geq J^* \bar{A}$

$$\therefore \boxed{J^* A = J^* \bar{A} = m\bar{A}}$$