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MATH 105 HW 4

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2 $f: \mathbb{R} \to [0, \infty)$

Pugh 25 (a) f is measurable $\Rightarrow$ $\mathcal{U}(f) \subset \mathbb{R}^2$ is measurable

We also know since f measurable, $\hat{\mathcal{U}}(f)$ is measurable with $m(\hat{\mathcal{U}}(f)) = m(\mathcal{U}(f))$

Note $\hat{\mathcal{U}}(f) = \mathcal{U}(f) + \text{Graph}(f)$

By finite additivity, $m(\hat{\mathcal{U}}(f)) = m(\mathcal{U}(f) + \text{Graph}(f))$

$= m(\mathcal{U}(f)) + m(\text{Graph}(f)) = m(\mathcal{U}(f))$

$\Rightarrow m(\text{Graph}(f)) = 0$ $\therefore$ $\boxed{\text{Graph}(f) \text{ is zero set.}}$

(b) No, let S be the set (nonmeasurable) from Tao (construction by rational cosets).
Then consider the function $f: \mathbb{R} \to \{0, 1\}$ where $f(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise.} \end{cases}$

Clearly, $m(\text{Graph}(f)) = 0$ since we just have to cover the lines $y = 0$ and $y = 1$ but f is not measurable.

(c) Transfinite induction: extension of induction to well-ordered sets

(d) Yes. If the measurability hypothesis wasn't there, $m(\text{Graph}(f)) > 0$ but for each slice across x, since $\text{Graph}(f) = \{(x, y) \mid y = f(x)\}$, each slice has measure 0 (one point). Hence theorem doesn't hold if the hypothesis is excluded

(e) Inner measure $m_*(A) = \sup\{\sum \text{vol}(\bar{B}_i)$ where $\bar{B}_i$ are disjoint closed boxes$\}$

Suppose on the contrary, that the inner measure of a graph > 0, say $= \varepsilon > 0$.
Then, $\exists$ a sequence of boxes $\bar{B}_1, \bar{B}_2, \ldots$ s.t. $\text{graph}(f) \supset \bigcup \bar{B}_i$ and $\sum \text{vol}(\bar{B}_i) \geq \frac{\varepsilon}{2} > 0$.

For each i, consider the dimension of the box parallel to the last coordinate. Since each slice across $\mathbb{R}^n \times \mathbb{R}$ w.r.t. $\mathbb{R}$ contains only 1 point (due to definition of $\text{graph}(f)$), $\text{vol}(\bar{B}_i) = 0$ $\forall i$ $\Rightarrow \sum \text{vol}(\bar{B}_i) = 0$ contradiction!

$\therefore$ $\boxed{\text{A graph can never have positive inner measure.}}$

(f) From (c), we have a function $f:[a,b] \to [0,\infty)$ whose graph is non measurable, in particular, $m^*(\text{graph}(f)) \neq 0$ (since all zero sets are measurable). $\therefore m^*(\text{graph}(f)) > 0$.

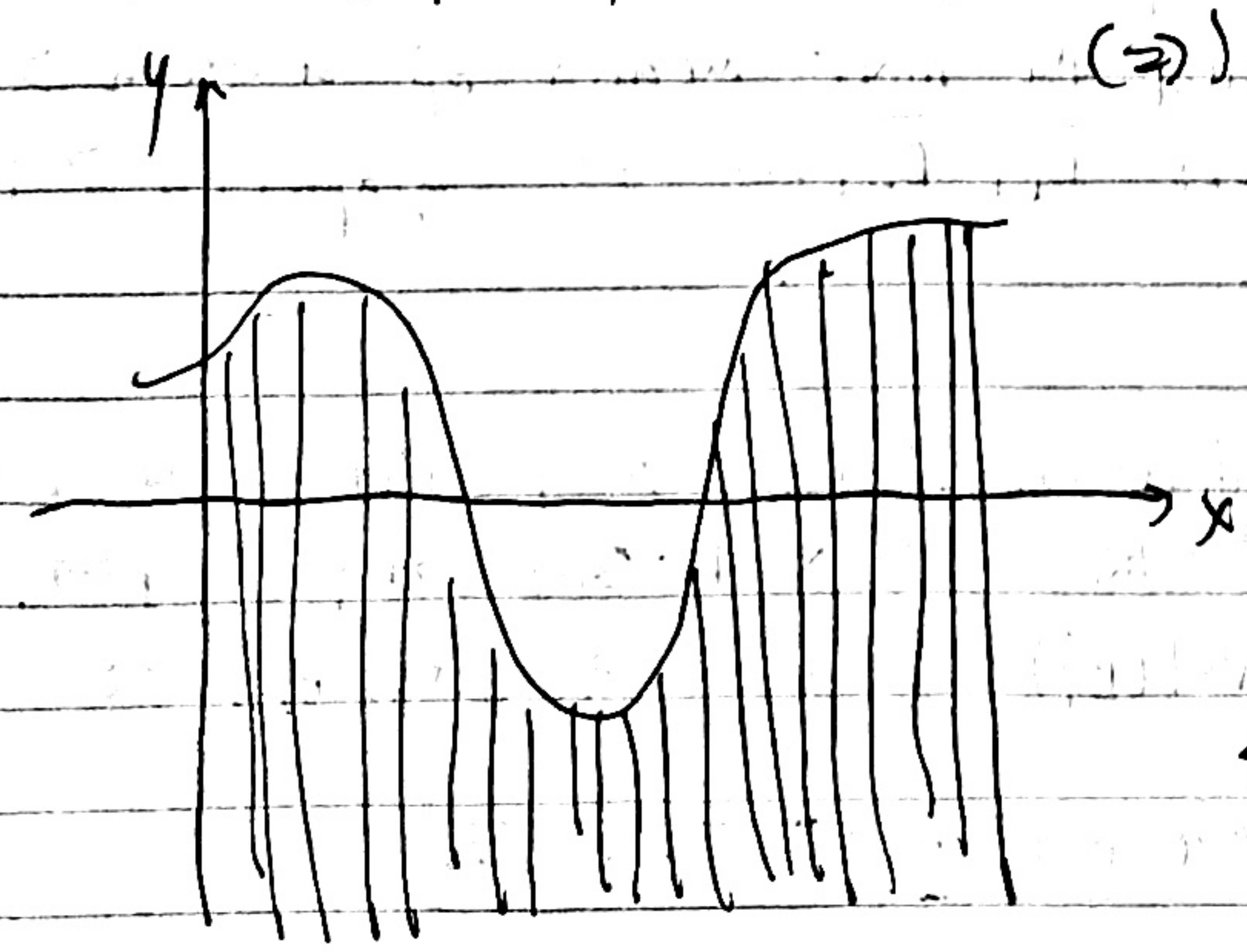
Hence, we can consider $f_n = f + n$ (i.e., $f_n(x) = f(x)+n$). $n \in \mathbb{Z}$. Consider the collection of all such functions f_n. We have a set in the subset of the plane $[a,b] \times \mathbb{R}$ with infinite outer measure.

~~Similarly, we can repeat this process for $[b+1, 2b-a+1]$, $[2b-a+2, 3b-2a+$ and so on. These are all disjoint subsets of the plane. Hence, in doing so, we partitioned the plane into un~~

Let S_r denote the set $\{(x,y) \mid y = f_n(x)+r\}$ where $r \in [0,1)$. Clearly, this is uncountable since it is indexed by r in the interval $[0,1)$. Each of these sets are disjoint and has infinite outer measure. Hence, we are done.

(g) In Exercise 19, it showed that it is possible to construct a mesisometry from a 1-dimensional interval I to a 2-dimensional square I^2. Due to the isometry, we can conclude that it is possible to ~~has~~ yield an example on $\mathbb{R}$ where there are an uncountably many disjoint subsets of $\mathbb{R}$, each having positive outer measure.

Pugh 28. 19 Total undergraph of $f:\mathbb{R}\to\mathbb{R}$: $\mathcal{U}f = \{(x,y) \mid y < f(x)\}$.



half plane

($\Rightarrow$) $\mathcal{U}f$ is measurable.

$\Rightarrow \mathcal{U}f \cap \{(x,y) \mid y \geq 0\}$ is also measurable since it is intersection of two measurable sets

$\Rightarrow$ $\boxed{\text{~~negative~~ positive part of } f \text{ is measurable}}$

$\mathcal{U}f \cup \{(x,y) \mid y \geq 0\}$ is measurable

$\Rightarrow (\mathcal{U}f \cup \{(x,y) \mid y \geq 0\})^c$ is measurable

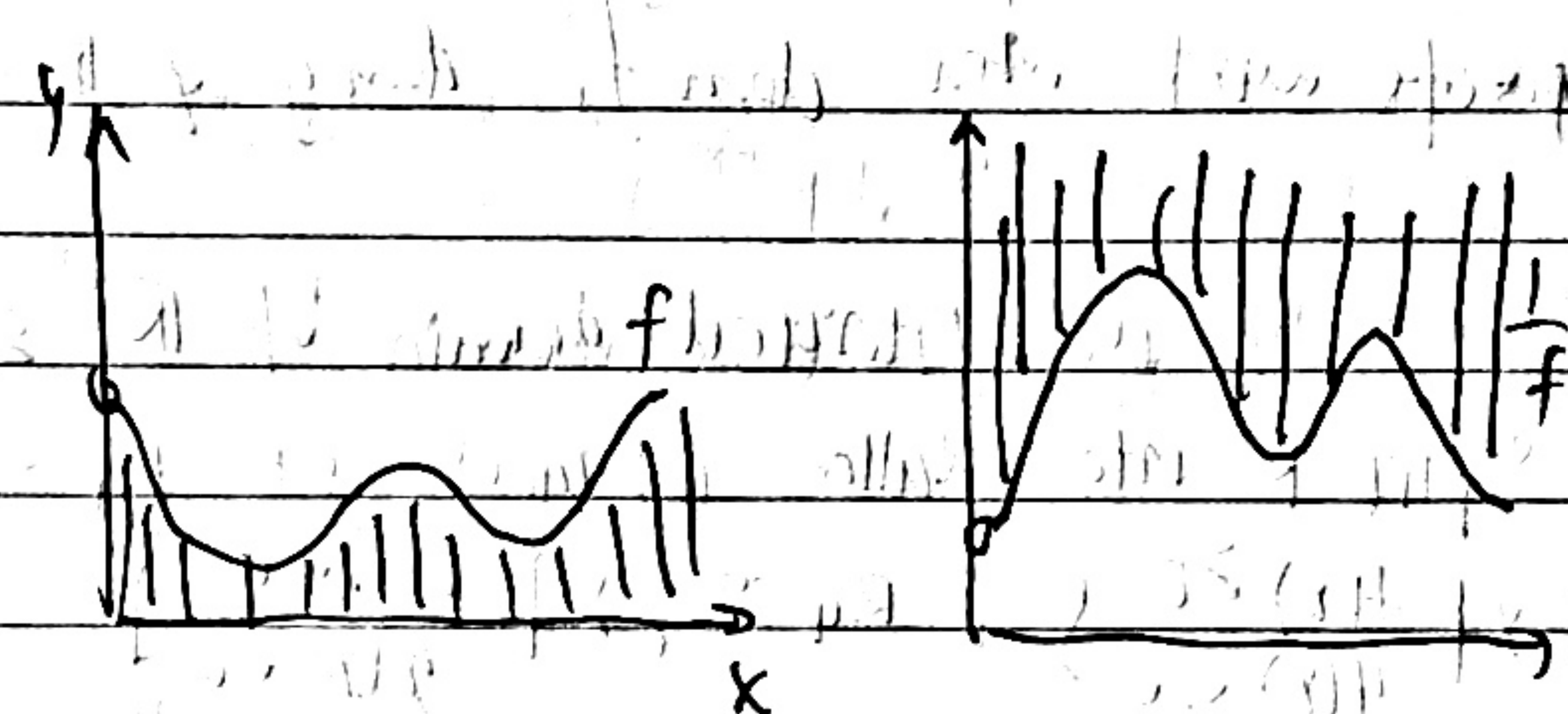
$\Rightarrow$ $\boxed{\text{negative part of } f \text{ is measurable.}}$

(⇐) By similar logic,

$$\left((\text{negative part})^c \cap \{(x,y) \mid y<0\} \right) \cup (\text{positive part})$$

is measurable since it is the complement, intersection and union of measurable sets $\Rightarrow$ $\boxed{\underline{\mathcal{U}} f \text{ is measurable}}$ as desired.

(b) $f: \mathbb{R} \to (0,\infty)$ is measurable $\Rightarrow$ $\mathcal{U}(f)$ is measurable.

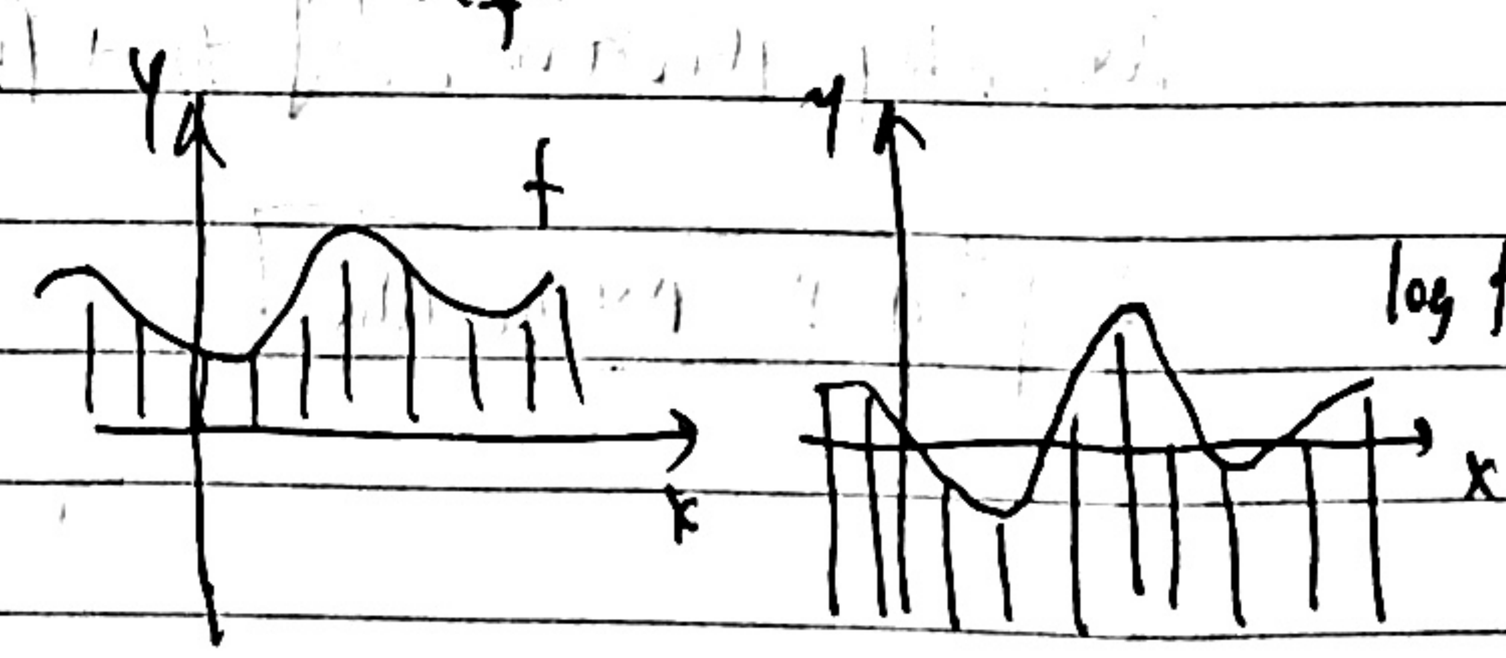


$(x,y) \in \mathcal{U}(f)$
$\Rightarrow y < f(x)$
$\Rightarrow \frac{1}{y} > \frac{1}{f(x)} \Rightarrow (x, \frac{1}{y}) \in \left(\hat{\mathcal{U}}(\frac{1}{f})\right)^c$

From exercise 23, a diffeomorphism preserves measurability
$\mathcal{U}(f)$ measurable $\Rightarrow \left(\hat{\mathcal{U}}(\frac{1}{f})\right)^c$ measurable $\Rightarrow \hat{\mathcal{U}}(\frac{1}{f})$ measurable
$\Rightarrow \mathcal{U}(\frac{1}{f})$ measurable

$\therefore$ $\boxed{\frac{1}{f} \text{ is measurable}}$

(c) $f, g: \mathbb{R} \to (0,\infty)$ are measurable.
Consider $T: (x,y) \to (x, \log y)$.



Then $(x,y) \in \mathcal{U}f \Rightarrow 0 < y < f(x)$
$\Rightarrow -\infty < \log y < \log f(x) \Rightarrow (x, \log y) \in \underline{\mathcal{U}}(\log f)$.

Similarly, under T, $\mathcal{U}g \to \underline{\mathcal{U}}(\log g)$.

Since T is a diffeomorphism, it also preserves measurability $\Rightarrow \underline{\mathcal{U}}(\log f)$ and $\underline{\mathcal{U}}(\log g)$ are measurable.

Claim: $\underline{\mathcal{U}}(\log fg) = \underline{\mathcal{U}}(\log f + \log g)$ is also measurable.
By the theorem, since $\log f, \log g: \mathbb{R} \to [0,\infty)$, then $\int \log f + \log g = \int \log f + \int \log g$, in particular $\int \log f + \log g = \int \log fg \Rightarrow \boxed{\log fg \text{ is measurable.}}$
$\Rightarrow \underline{\mathcal{U}}(\log fg)$ measurable

Finally, $T^{-1}: (x,y) \to (x, e^y)$ is a diffeomorphism hence preserves measurability
$\underline{\mathcal{U}} \log fg$ measurable $\Rightarrow (x,y) \in \underline{\mathcal{U}} \log fg$) next page

$\underline{\mathcal{U}}\, fg^{\log}$ measurable $\Rightarrow$ $(x, y) \in \underline{\mathcal{U}}\, fg^{\log}$ $\Rightarrow$ $y < \log(fg(x))$

$\Rightarrow$ as $e^y < f(x)\, g(x)$

$\Rightarrow$ $(x, e^y) \in \mathcal{U}(f(x)\, g(x))$

Hence $\mathcal{U}(f(x) g(x))$ is measurable $\Rightarrow$ $\boxed{fg \text{ is measurable}}$

(d) In my proofs for (a), (b), (c) I did not require any condition on x. I only required $y \in \mathbb{R}$. Hence, the proofs work when $\operatorname{dom} f$, $\operatorname{dom} g \neq \mathbb{R}$.

(e) Let $f, g : \mathbb{R} \to (-\infty, \infty)$. Let R be the restricted domain of $\mathbb{R}$ s.t. $\forall x \in R$, $f(x) \neq 0$, $g(x) \neq 0$. Further split R into smaller domains s.t. $R_1 = \left\{ x \,\middle|\, \begin{matrix} f(x) > 0 \\ g(x) > 0 \end{matrix} \right\}$, $R_2 = \left\{ x \,\middle|\, \begin{matrix} f(x) < 0 \\ g(x) > 0 \end{matrix} \right\}$, $R_3 = \left\{ x \,\middle|\, \begin{matrix} f(x) > 0 \\ g(x) < 0 \end{matrix} \right\}$, $R_4 = \left\{ x \,\middle|\, \begin{matrix} f(x) < 0 \\ g(x) < 0 \end{matrix} \right\}$

Then, from (c), R_1, R_2, R_3, R_4 are all measurable by considering the combination of $\pm|f|, \pm|g|$. Let fg_i correspond to R_i

Then, by theorem, $\int fg_1 + fg_2 + fg_3 + fg_4 = \int fg$ is measurable.

$\therefore$ $\boxed{fg \text{ is measurable}}$