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MATH 105 HW 5

8.2.7 Denote by Ω_q to be the set $\{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p}\}$ for a fixed value q. Since $p > 2$, for large enough q, we can get $\frac{c}{q^p} < \frac{1}{2q}$.

Hence, suffices to consider $0 \leq a \leq q$, because for values of a outside this range, $x \in [0,1]$, $|x - \frac{a}{q}| \geq \frac{1}{2q} > \frac{c}{q^p}$, so there will be no solution of x in $[0,1]$.

Hence, the intervals in Ω_q are: $[\frac{1}{q} - \frac{c}{q^p}, \frac{1}{q} + \frac{c}{q^p}], [\frac{2}{q} - \frac{c}{q^p}, \frac{2}{q} + \frac{c}{q^p}], \ldots$ $[1 - \frac{c}{q^p}, 1]$, all of which are disjoint since $(\frac{c}{q^p} < \frac{1}{2q})$.

Hence, the measure of $\Omega_q = \boxed{(2q-1)\frac{c}{q^p}}$.

Since $\sum_{q=1}^{\infty} (2q-1)\frac{c}{q^p}$ converges $\forall q$ (by the p-test since $p>2$), the tail of $\sum_{q=1}^{\infty} m(\Omega_q)$ converges (i.e. $< \infty$). By Borel Cantelli lemma, the set $\{x : x \in \Omega_q \text{ for infinitely many } q\}$ has measure $\boxed{0}$.

8.29. $n \in \mathbb{N}$. $f_n : \mathbb{R} \to [0,\infty)$ nonnegative, measurable s.t. $\int_{\mathbb{R}} f_n \leq \frac{1}{4^n}$.

Firstly, note that $m(\{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon \cdot 2^n}\}) \leq \frac{\varepsilon}{2^n}$. Suppose otherwise, then, since f_n is measurable, the preimage of $(\frac{1}{\varepsilon \cdot 2^n}, \infty)$ is measurable (say E). Then, by our assumption $m(E) > \frac{\varepsilon}{2^n}$. Let S be a simple fn that is χ_E. Then S minorizes f_n, so $\int_{\mathbb{R}} f_n \geq \int_{\mathbb{R}} S > \frac{\varepsilon}{2^n} \cdot \frac{1}{\varepsilon \cdot 2^n} = \frac{1}{4^n}$ (a contradiction!)

Hence $\boxed{m(\{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon \cdot 2^n}\}) \leq \frac{\varepsilon}{2^n}}$.

Now, consider the union of all these sets. All of these sets are measurable, since they are the premimage of open sets. Since $\sum_{n=1}^{\infty} m(\{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon \cdot 2^n}\}) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon < \infty$, by Borel Cantelli, the set of x that ~~does not~~ belongs to infinitely many such sets $m(\{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon \cdot 2^n}\})$ has measure 0. But these are exactly the x that does not converge pointwise to 0 under f_n (the complement of this set are all x that belongs to finitely such set, i.e. $\exists N$ s.t. $n \geq N \Rightarrow f_n(x) \leq \frac{1}{\varepsilon \cdot 2^n} \; \forall n$
$\Rightarrow \lim_{n \to \infty} f_n(x) \leq \lim_{n \to \infty} \frac{1}{\varepsilon \cdot 2^n} = 0$)

Hence, the set of x that does not converge pointwise to 0 is a measure 0 set. Consequently, $\forall \varepsilon > 0$, $\exists E$ with $m(E) \leq \varepsilon$ s.t. $f_n(x)$ converges pointwise to 0 for $x \in \mathbb{R} \setminus E$ (just contain Z, the set of x that does not converge pointwise to 0). ~~(Actually, I just realized we can just eliminate the union, which has measure at most ε)~~

8.2.10 $\forall n \in \mathbb{N}$, $f_n : [0,1] \to [0,\infty)$ nonnegative measurable s.t. $f_n \xrightarrow{\text{pointwise}} 0$.

Denote $S_N = \{x \mid f_n(x) \leq \delta \ \forall n > N\}$.
Then, note that for a fixed δ, $\lim\limits_{n\to\infty} S_n = [0,1]$, since all points converge pointwise to ~~infinity~~ 0.
Also note that $(S_n)_n$ are strictly increasing sets and are measurable (because
$S_N = \left(f_{N+1}^{-1}((\delta,\infty))\right)^c \cap \left(f_{N+2}^{-1}((\delta,\infty))\right)^c \cap \dots$ (complement taken in $[0,1]$)

which is a countable intersection of premages of measurable functions $\Rightarrow$ measurable)

Hence, by upward limit continuity, $\lim\limits_{n\to\infty} m(S_n) = m([0,1]) = 1$

$\Rightarrow \forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, $m(S_n) > 1 - \varepsilon$

Note S_n and S_n^c are disjoint, so $m(S_n) + m(S_n^c) = 1 \Rightarrow m(S_n^c) < \varepsilon$
Hence $m(\{x \in [0,1] : \underset{\text{at least once in } n>N}{f_n(x) > \delta}\}) < \varepsilon$.

In particular,

$$m(\{x \in [0,1] : f_n(x) \geq \delta\}) \leq m(\{x \in [0,1] : \underset{\text{at least once}}{f_n(x) > \delta}\}) < \varepsilon$$

Since this holds $\forall \delta, \varepsilon$, we conclude can find N s.t.

$$\boxed{m(\{x \in [0,1] : f_n(x) > \tfrac{1}{m}\}) \leq \tfrac{\varepsilon}{2^m} \text{ for any } m.}$$

Hence, let $\varepsilon > 0$, choose $N = \max(N_1, N_2, \dots N_{\frac{1}{\varepsilon}})$ where N_k denotes the N chosen s.t. $m(\{x \in [0,1] : f_n(x) > \frac{1}{k}\}) \leq \frac{\varepsilon}{2^k}$
Then, let $E = \bigcup\limits_{i=1}^{\frac{1}{\varepsilon}} \{x \in [0,1] : f_n(x) > \frac{1}{i}\}$. Thus, on $[0,1] \setminus E$, $\forall m, n > N$,
$\Rightarrow m(E) \leq \sum \frac{\varepsilon}{2^i} = \varepsilon$

$|f_m(x) - f_n(x)| < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$. Hence $f_n(x)$ converges uniformly to zero on $[0,1] \setminus E$.

If $[0,1]$ is replaced by $\mathbb{R}$, then the claim will not hold.
Consider $f_n = \chi_{[n, n+1)}$ for $n \geq 1$

Then $(f_n)_n$ converges pointwise to 0, but $(f_n)_n$ will not converge uniformly to 0 on any $\mathbb{R} \setminus E$ with $m(E) < \frac{1}{2}$, because for any N, $\exists$ interval $[N+1, N+2)$ and therefore f_{N+1} that gives value 1 on that interval.