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MATH 105 HW 6

8.3.2 Ω: measurable, $f: \Omega \to \mathbb{R}$, $g: \Omega \to \mathbb{R}$ absolutely integrable (i.e. $\int_\Omega |f| < \infty$, $\int_\Omega |g| < \infty$)

(a) Firstly, note cf is $\boxed{\text{absolutely integrable}}$, since

$$\int_\Omega |cf| = \int_\Omega |c||f| \underset{\text{Proposition 8.2.6 (b)}}{=} |c| \int_\Omega |f| < \infty \quad \text{since } \int_\Omega |f| < \infty$$

Case #1: $c > 0$. Write $f = f^+ - f^-$ where $f^+ = \max(f, 0)$, $f^- = -\min(0, f^-)$

Both f^+ and f^- are measurable

Then: $cf = c(f^+ - f^-) = cf^+ - cf^-$ where $cf^+, cf^-: \Omega \to [0, \infty)$

Hence, $\int_\Omega cf = \int_\Omega cf^+ - \int_\Omega cf^-$ (since $cf^+ = \max(0, c\cdot f)$, $cf^- = -\min(0, c\cdot f)$)

$= c\int_\Omega f^+ - c\int_\Omega f^-$ (Proposition 8.2.6 (b))

$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c \cdot \int_\Omega f \Rightarrow \boxed{\int_\Omega cf = c\int_\Omega f}$ for $c > 0$.

Case #2: $c = 0$. There's nothing to prove here since $\int_\Omega c\cdot f = 0$ and $c \cdot \int_\Omega f = 0$.

$\Rightarrow \boxed{\int_\Omega c\cdot f = c\int_\Omega f = 0}$

Case #3: $c < 0$.

Then $cf = c(f^+ - f^-) = cf^+ - cf^- = -cf^- + cf^+$

$= -cf^- - (-cf^+)$

Note $\max(0, cf) = -cf^-$

$-\min(0, cf) = -cf^+$

by proposition 8.2.6(b) since $-c > 0$.

$\therefore \int_\Omega cf = \int_\Omega -cf^- - \int_\Omega -cf^+ = -c\int_\Omega f^- - (-c)\int_\Omega f^+$

$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c\int_\Omega f.$

$\therefore \boxed{\int_\Omega cf = c\int_\Omega f}$ for $c < 0$ as desired

$\therefore \boxed{\int_\Omega cf = c\int_\Omega f.}$

(b) Firstly, I will show $f+g$ is absolutely integrable: Lemma 8.2.10, since $|f|, |g|: \Omega \to [0, \infty]$. ~~Proposition~~

$$\int_\Omega |f+g| \leq \int_\Omega |f| + |g| = \int_\Omega |f| + \int_\Omega |g| < \infty \quad \text{since both } \int_\Omega |f|, \int_\Omega |g| < \infty$$

(Proposition 8.2.6 (c) since $|f+g| \leq |f| + |g|$) $\quad \therefore \boxed{f+g \text{ absolutely integrable}}$

8.3.2 (b) continued.

Note $\max(f+g, 0) = f^+ + g^+$

$-\min(f+g, 0) = f^- + g^-$

$$\Rightarrow \int_\Omega f+g = \int_\Omega f^+ + g^+ - \int_\Omega f^- + g^-$$

$$= \int_\Omega f^+ + \int_\Omega g^+ - \int_\Omega f^- - \int_\Omega g^- \quad (\text{by } \cancel{\text{Proposition}} \text{ Lemma } 8.2.10)$$

$$= \left(\int_\Omega f^+ - \int_\Omega f^-\right) + \left(\int_\Omega g^+ - \int_\Omega g^-\right) = \int_\Omega f + \int_\Omega g$$

$$\therefore \boxed{\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g}$$

(c) Note $g-f$ is absolutely integrable. This is because f absolutely integrable

$\Rightarrow -f$ absolutely integrable

$\Rightarrow g-f$ absolutely integrable by (b)

Hence, $\int_\Omega g = \int_\Omega (g-f) + \int_\Omega f$ (as obtained from part b).

$\Rightarrow \int_\Omega g - \int_\Omega f = \int_\Omega g-f$. Note $g-f \geq 0 \ \forall x \in \Omega$.

Hence $\int g-f \geq 0$ (also by Proposition 8.2.6 (a))

$\Rightarrow \int_\Omega g - \int_\Omega f \geq 0 \Rightarrow \boxed{\int_\Omega f \leq \int_\Omega g}$ as desired

(d) $f(x) = g(x)$ for almost every $x \in \Omega$.

Define $h = g-f$. Then $h(x) = 0$ almost everywhere $x \in \Omega$. $\Rightarrow$ both h^+ and h^- are 0 for almost every x.

By above logic, h is absolutely integrable.

$$\int_\Omega h = \int_\Omega h^+ - \int_\Omega h^- = 0 - 0 = 0 \quad (\text{by Proposition } 8.2.6(a))$$

$$\Rightarrow \int_\Omega g = \int_\Omega f + \int_\Omega g-f = \int_\Omega f + \int_\Omega h = \int_\Omega f$$

$\therefore \boxed{\int_\Omega g = \int_\Omega f.}$ as desired.

8.3.3. $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ absolutely integrable i.e. $\int_\Omega |f| < \infty$, $\int_\Omega |g| < \infty$.

$f(x) \leq g(x) \ \forall x \in \mathbb{R}$. $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$.

8.3.3 continued

From the works of 8.3.2, we know $g-f$ is also absolutely integrable and

$$\int_{\Omega} g = \int_{\Omega} f + \int_{\Omega} g-f$$

$g-f : \mathbb{R} \to [0,\infty)$

Since $g(x) \geq f(x)$ $\forall x \in \mathbb{R}$, $g(x)-f(x) \geq 0 \Rightarrow \int_{\Omega} g-f \geq 0$.

We also have $\int_{\mathbb{R}} g = \int_{\mathbb{R}} f \Rightarrow \int_{\mathbb{R}} g-f = 0$.

But by Proposition 8.2.6, $\int_{\mathbb{R}} g-f = 0 \Rightarrow (g-f)(x) = 0$ for almost every $x \in \mathbb{R}$

$\Rightarrow g(x) = f(x)$ for almost every $x \in \mathbb{R}$.

$\therefore$ $\boxed{f(x) = g(x) \text{ for almost every } x \in \mathbb{R}}$, as desired.

3. Refer to Student Area

4. Ok.