MATH 105 HW 6

8.3.2 Ω: measurable, $f:\Omega\to\mathbb{R}$, $g:\Omega\to\mathbb{R}$ absolutely integrable (i.e. $\int_\Omega |f|<\infty$, $\int_\Omega |g|<\infty$)

(a) Firstly, note cf is $\boxed{\text{absolutely integrable}}$, since

$$\int_\Omega |cf| = \int_\Omega |c||f| \underset{\text{Proposition 8.2.6 (b)}}{=} |c|\int_\Omega |f| < \infty \quad \text{since } \int_\Omega |f|<\infty$$

Case #1: $c>0$. Write $f=f^+-f^-$ where $f^+=\max(f,0)$, $f^-=-\min(0,f^-)$

Both f^+ and f^- are measurable

Then: $cf = c(f^+-f^-) = cf^+ - cf^-$ where $cf^+, cf^-:\Omega\to[0,\infty)$

Hence, $\int_\Omega cf = \int_\Omega cf^+ - \int_\Omega cf^-$ (since $cf^+=\max(0,c\cdot f)$, $cf^- = -\min(0,c\cdot f)$)

$= c\int_\Omega f^+ - c\int_\Omega f^-$ (Proposition 8.2.6(b))

$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c\cdot\int_\Omega f \Rightarrow \boxed{\int_\Omega cf = c\int_\Omega f}$ for $c>0$.

Case #2: $c=0$. There's nothing to prove here since $\int_\Omega c\cdot f = 0$ and $c\cdot\int_\Omega f = 0$.

$\Rightarrow \boxed{\int_\Omega c\cdot f = c\int_\Omega f = 0}$

Case #3: $c<0$.

Then $cf = c(f^+-f^-) = cf^+ - cf^- = -cf^- + cf^+$

$= -cf^- - (-cf^+)$

Note $\max(0, cf) = -cf^-$

$-\min(0, cf) = -cf^+$

by proposition 8.2.6(b) since $-c>0$.

$\therefore \int_\Omega cf = \int_\Omega -cf^- - \int_\Omega -cf^+ = -c\int_\Omega f^- - (-c)\int_\Omega f^+$

$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c\int_\Omega f.$

$\therefore \boxed{\int_\Omega cf = c\int_\Omega f}$ for $c<0$ as desired

$\therefore \boxed{\int_\Omega cf = c\int_\Omega f.}$

(b) Firstly, I will show $f+g$ is absolutely integrable: Lemma 8.2.10, since $|f|, |g|:\Omega\to[0,\infty]$. ~~Proposition~~

$\int_\Omega |f+g| \le \int_\Omega |f|+|g| = \int_\Omega |f| + \int_\Omega |g| < \infty$ since both $\int_\Omega |f|, \int_\Omega |g| <\infty$

(Proposition 8.2.6 (c) since $|f+g|\le |f|+|g|$)

$\therefore \boxed{f+g \text{ absolutely integrable}}$

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8.3.2 (b) continued.

We know that $\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-$ and $\int_\Omega g = \int_\Omega g^+ - \int_\Omega g^-$

Hence, suffices to show
$$\int_\Omega f+g \equiv \int_\Omega (f+g)^+ - \int_\Omega (f+g)^- \overset{?}{=} \int f^+ - \int_\Omega f^- + \int_\Omega g^+ - \int_\Omega g^-$$

$$\Rightarrow \int_\Omega (f+g)^+ + f^- + g^- = \int_\Omega (f+g)^- + f^+ + g^+$$

We note that at each point $x \in \Omega$, we have the following cases:

Case #1: $f(x) \geq 0$, $g(x) \geq 0 \Rightarrow (f+g)(x) \geq 0$

$\therefore f^+(x) = f(x)$, $f^-(x) = 0$, $g^+(x) = g(x)$, $g^-(x) = 0$, $(f+g)^+(x) = (f+g)(x) = f(x)+g(x)$, $(f+g)^-(x) = 0$.

$\therefore$ ~~LHS =~~ $\cancel{\int_\Omega} (f+g)^+ + f^- + g^- = \cancel{\int_\Omega}(f+g)(x) = f(x) + g(x) = (f+g)^-(x) + f^+(x) + g^+(x)$

Case #2: $f(x) \geq 0$, $g(x) < 0$.

Subcase #1: If $(f+g)(x) \geq 0$, then $f^+(x) = f(x)$, $f^-(x) = 0$, $g^+(x) = 0$, $g^-(x) = -g(x)$, $(f+g)^+(x) = f(x) + g(x)$, $(f+g)^-(x) = 0$

$\therefore$ LHS $= (f+g)^+(x) + f^-(x) + g^-(x) = f(x) + g(x) + (-g(x)) = \cancel{f(x) - 2g(x)}$ $f(x)$.

RHS $= (f+g)^- + f^+ + g^+ = f(x)$ $\therefore$ LHS = RHS.

Subcase #2: If $(f+g)(x) < 0$ then $f^+(x) = f(x)$, $f^-(x) = 0$, $g^+(x) = 0$, $g^-(x) = -g(x)$, $(f+g)^+(x) = 0$, $(f+g)^-(x) = -f(x) - g(x)$

$\therefore$ LHS $= (f+g)^+(x) + f^-(x) + g^-(x) = -g(x)$.

RHS $= (f+g)^-(x) + f^+(x) + g^+(x) = -f(x) - g(x) + f(x) = -g(x)$ $\therefore$ LHS = RHS

Case #3: $f(x) < 0$, $g(x) \geq 0$ (Similar to case 2 by symmetry).

Case #4: $f(x) < 0$, $g(x) < 0 \Rightarrow f^+(x) = 0$, $f^-(x) = -f(x)$, $g^+(x) = 0$, $g^-(x) = -g(x)$, $(f+g)^+(x) = 0$, $(f+g)^-(x) = -g(x) - f(x)$.

LHS $= (f+g)^+(x) + f^-(x) + g^-(x) = -f(x) - g(x)$.

RHS $= (f+g)^-(x) + f^+(x) + g^+(x) = -g(x) - f(x)$ $\therefore \boxed{\text{LHS} = \text{RHS}}$

Hence, since $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$ everywhere and they are all nonnegative functions, we have: $\int_\Omega (f+g)^+ + f^- + g^- = \int_\Omega (f+g)^- + f^+ + g^+$

Consequently, $\boxed{\int_\Omega f+g = \int_\Omega f + \int_\Omega g}$ as desired.

(c) Note that $g-f$ is absolutely integrable. This is because f absolutely integrable
$\Rightarrow -f$ absolutely integrable
$\Rightarrow g-f$ absolutely integrable by (b)

Hence, $\int_\Omega g = \int_\Omega (g-f) + \int_\Omega f$ (as obtained from (b))

$\Rightarrow \int_\Omega g - \int_\Omega f = \int_\Omega g-f$ Note $g-f \geq 0 \ \forall x \in \Omega$
Hence $\int g-f \geq 0$ (also by Proposition 8.2.6 (a))

$\Rightarrow \int_\Omega g - \int_\Omega f \geq 0 \Rightarrow \boxed{\int_\Omega f \leq \int_\Omega g}$ as desired.

(d) $f(x) = g(x)$ for almost every $x \in \Omega$.
Define $h = g - f$. Then $h(x) = 0$ almost everywhere $x \in \Omega \Rightarrow$ both h^+ and h^- are 0 almost everywhere

Hence, h is absolutely integrable, since $|h|$ is 0 almost everywhere

$\Rightarrow \int_\Omega h = \int_\Omega h^+ - \int_\Omega h^- = 0 - 0 = 0$ (from Proposition 8.2.6 (a))

$\Rightarrow \int_\Omega g = \cancel{\int_\Omega h + f} \int_\Omega f + \int_\Omega g-f = \int_\Omega f + \int_\Omega h = \int_\Omega f$

$\therefore \boxed{\int_\Omega g = \int_\Omega f \text{ as desired}}$

8.2.3. $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ absolutely integrable i.e. $\int_\Omega |f| < \infty$, $\int_\Omega |g| < \infty$
~~$f(x) \leq g(x) \ \forall x \in \mathbb{R}$, $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$~~

next page.

8.3.3 continued

From the works of 8.3.2, we know $g-f$ is also absolutely integrable and

$$\int_{\mathbb{R}} g = \int_{\mathbb{R}} f + \int_{\mathbb{R}} g-f$$

$g-f : \mathbb{R} \to [0,\infty)$

Since $g(x) \geq f(x)\ \forall x \in \mathbb{R}$, $g(x) - f(x) \geq 0 \Rightarrow \int_{\mathbb{R}} g-f \geq 0$.

We also have $\int_{\mathbb{R}} g = \int_{\mathbb{R}} f \Rightarrow \int_{\mathbb{R}} g-f = 0$.

But by Proposition 8.2.6, $\int_{\mathbb{R}} g-f = 0 \Rightarrow (g-f)(x) = 0$ for almost every $x \in \mathbb{R}$

$\Rightarrow g(x) = f(x)$ for almost every $x \in \mathbb{R}$.

$\therefore \boxed{f(x) = g(x) \text{ for almost every } x \in \mathbb{R}}$, as desired.

3. Refer to Student Area

4. Ok.