MATH 105 HW 7

Problem 1. 6.39

f, g measurable $\to$ $\boxed{fg \text{ measureable}}$ by Exercise 28.
f^2, g^2 integrable $\Rightarrow \int f^2, \int g^2 < \infty \Rightarrow \int f^2 + g^2 < \infty$.

Note that $(f-g)^2 \geq 0 \Rightarrow f^2 + g^2 \geq 2fg \Rightarrow \frac{f^2+g^2}{2} \geq fg$.
Hence,
$$\int fg \leq \int \frac{f^2+g^2}{2} = \frac{1}{2}\int f^2 + \frac{1}{2}\int g^2 < \infty \Rightarrow \boxed{fg \text{ also integrable}}$$

By Proposition 6.38, the set of measurable functions $f: \mathbb{R} \to \mathbb{R}$ is a vector space the set of integrable functions is a subspace, and integral is a linear map from the set of integrable functions to $\mathbb{R}^n$.

Define the inner product $\langle f, g \rangle = \int fg$. It satisfies positive definiteness since $\langle f, f \rangle = \int f^2 \geq 0$ with equality only when $f^2 = 0$ a.e. i.e. $f = 0$ a.e. (we consider two functions to be equivalent if $f = g$ a.e.) Also, it satisfies symmetry, as $\langle f, g \rangle = \int fg = \int gf = \langle g, f \rangle$. Lastly, it is bilinear, $\langle cf + h, g \rangle = \int (cf+h) g = \int cfg + hg = \int cfg + \int hg = c\int fg + \int hg = c\langle f, g \rangle + \langle h, g \rangle$

Hence, by Cauchy-Schwarz inequality $|\langle f, g \rangle|^2 \leq \langle f, f \rangle \langle g, g \rangle$
$$\Rightarrow \boxed{\left(\int fg\right)^2 \leq \int f^2 \int g^2}$$
$$\Rightarrow \boxed{\int fg \leq \sqrt{\int f^2 \int g^2}} \text{ as desired.}$$

Problem 2
6.48

$H:[0,1]\to[0,1]$

$\hat{H}(x+n)=H(x)+n$ for $n\in\mathbb{Z}$.

$H_k(x)=\hat{H}(3^k x)$ $\quad J(x)=\sum_{k=0}^{\infty}\frac{H_k(x)}{4^k}$.

Firstly, I will show J is continuous. We are given that H is continuous, therefore, $\hat{H}$ is continuous since it would be continuous $\forall x\notin\mathbb{Z}$ and at $\forall x\in\mathbb{Z}$, the left and right limits are both equal to x. Hence H_k is continuous $\forall k$. $J(x)$ is the uniform limit of continuous functions, since $J(x)=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{H_k(x)}{4^k}\Rightarrow$ $\boxed{J \text{ is continuous}}$.

Next I will show J is strictly increasing

$$J(x)=\sum_{k=0}^{\infty}\frac{H_k(x)}{4^k}=\sum_{k=0}^{\infty}\frac{\hat{H}(3^k x)}{4^k}=\sum_{k=0}^{\infty}\frac{\hat{H}(\{3^k x\})+\lfloor 3^k x\rfloor}{4^k}$$

Here, $\{x\}=x-\lfloor x\rfloor$ denotes the fractional part of x

$$=\sum_{k=0}^{\infty}\frac{H(\{3^k x\})+\lfloor 3^k x\rfloor}{4^k}$$

Let's restrict our attention to the form $J(x)=\sum_{k=0}^{\infty}\frac{\hat{H}(3^k x)}{4^k}$.

Note $\hat{H}(x)$ is an ~~strict~~ ~~inc~~ nondecreasing function (inherited this from H)

Also, let $x<y$, then $\exists k\in\mathbb{Z}$ s.t. $y-x>\frac{1}{3^k}$.

Then $$\frac{\hat{H}(3^k y)}{4^k}=\frac{\hat{H}(3^k(y-x)+3^k x)}{4^k}\geq\frac{\hat{H}(3^k x+1)}{4^k}=\frac{\hat{H}(3^k x)+1}{4^k}>\frac{\hat{H}(3^k x)}{4^k}$$

$\therefore J(y)>J(x)$ since the other terms are at least as big

$\therefore$ $\boxed{J \text{ is strictly increasing.}}$

Now, let's discover some properties of J: Firstly, J satisfies $J(x+1)=J(x)+4$.

Proof: $$J(x+1)=\sum_{k=0}^{\infty}\frac{H(\{3^k(x+1)\})+\lfloor 3^k(x+1)\rfloor}{4^k}=\sum_{k=0}^{\infty}\frac{H(\{3^k x+3^k\})+\lfloor 3^k x+3^k\rfloor}{4^k}$$

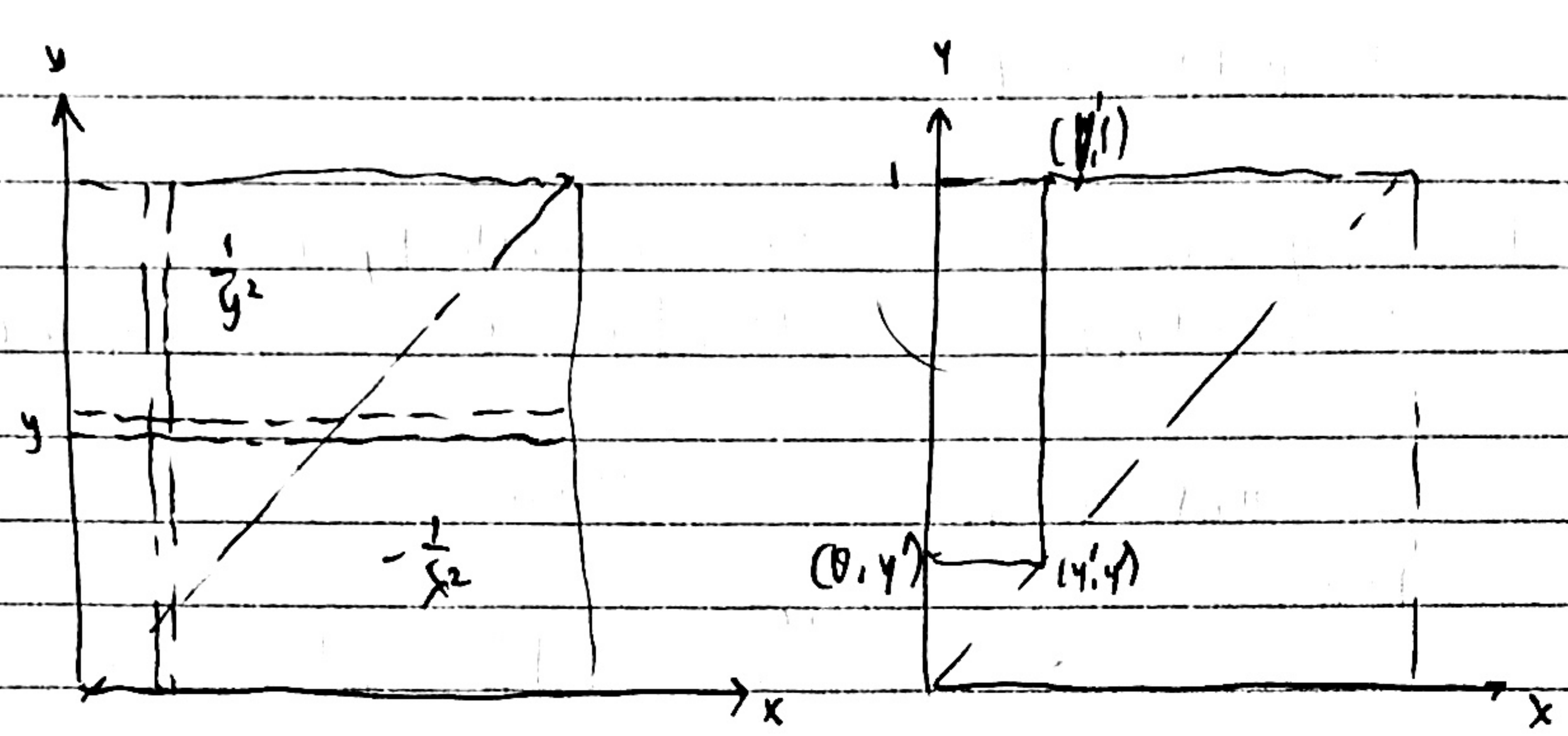
$$=\sum_{k=0}^{\infty}\frac{H(\{3^k x\})+\lfloor 3^k x\rfloor+3^k}{4^k}=4+\sum_{k=0}^{\infty}\frac{H(\{3^k x\})+\lfloor 3^k x\rfloor}{4^k}$$

Note $\{3^k x+3^k\}=\{3^k x\}$ since $3^k\in\mathbb{Z}$.

$$=4+J(x)$$

Problem 3.
6.53

$f: \mathbb{R}^2 \to \mathbb{R}$



(a) $\int f(x,y)\,dx = \int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx = \left[\frac{1}{y^2}x\right]_0^y + \left[\frac{1}{x}\right]_y^1$

$= \frac{1}{y} + 1 - \frac{1}{y} = \boxed{1}$ finite.

$\int f(x,y)\,dy = \int_0^x -\frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dy = \left[-\frac{1}{x^2}y\right]_0^x + \left[-\frac{1}{y}\right]_x^1$

$= -\frac{1}{x} - 1 + \frac{1}{x} = \boxed{-1}$ finite.

~~$\iint f(x,y)\,dA$. For simplicity, consider the region (rectangle bounded by $(0,y)$ and $(y,1)$)~~
~~The measure of the under graph can be computed by~~

In particular, I will just show the upper integral does not exist.

$$\iint_{\nabla} f\,dA = \int_0^1 \int_0^y \frac{1}{y^2}\,dx\,dy = \int_0^1 \left[\cancel{-\frac{1}{y}}\right] \frac{1}{y}\,dy = \left[\ln y\right]_0^1 = \infty.$$

Meanwhile, the integral of the lower triangle is similarly $-\infty$. Hence double integral $\boxed{\text{does not exist.}}$

(b) The only reason why it does not contradict is because 43 assumes $f: \mathbb{R}^2 \to [0,\infty)$. Here, our function is both negative and positive. If in particular, instead of $-\frac{1}{x^2}$, we have $\frac{1}{x^2}$, then all three integrals ~~with~~ will be equal ~~and~~ and $= \infty$.

Hence, does not contradict 43.

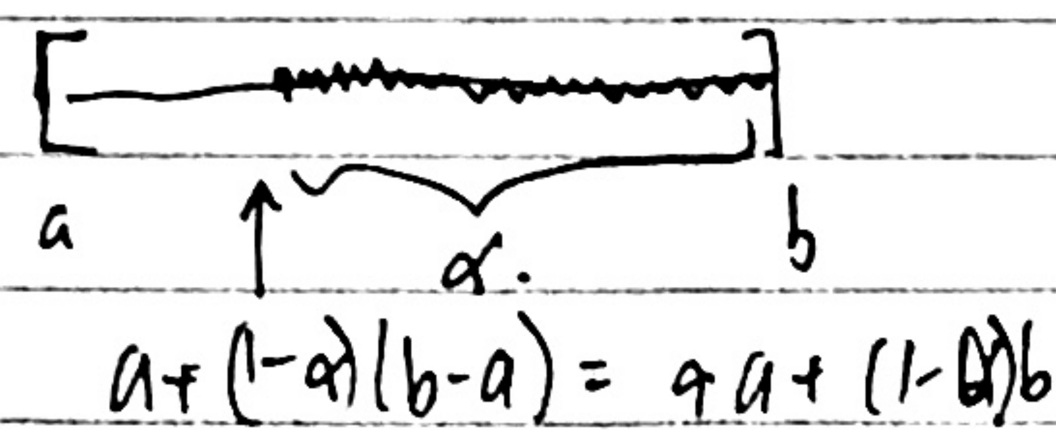
Problem 4
6.58

(a) The density of E at p is defined as $d(p,E) = \lim_{Q \downarrow p} \frac{m(E \cap Q)}{m(Q)}$ where the cubes need not be centered at p. In particular, if p is a density point of E, then p automatically satisfies the condition for being a balanced density point (a more restrictive definition). Hence, almost every point of E is a balanced density point.

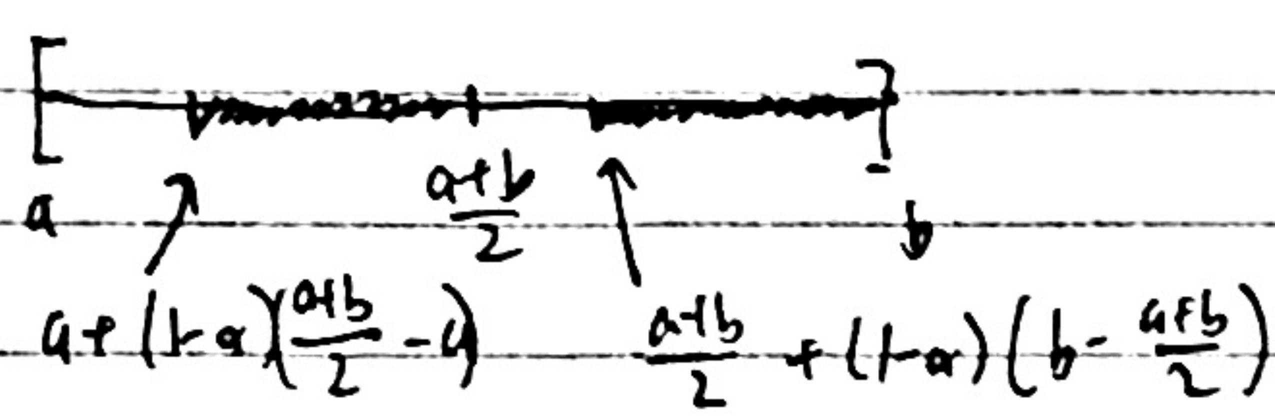
Let $\alpha \in [0,1]$

(b) Consider a function $E_k([a,b])$ that performs the following operation on the interval $[a,b]$:

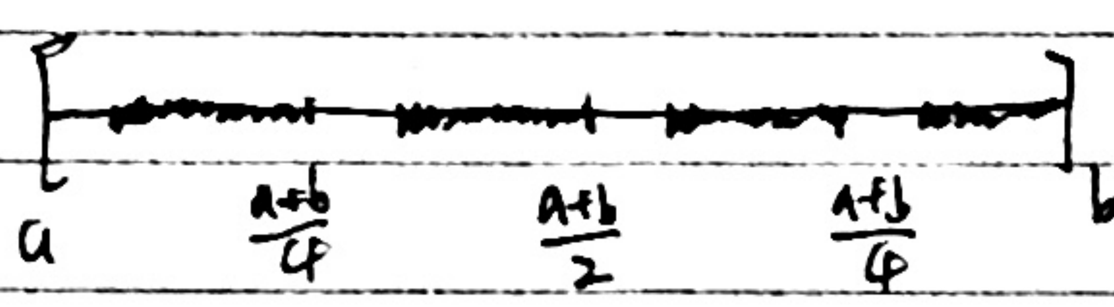
$E_1([a,b])$:



$E_2([a,b])$:

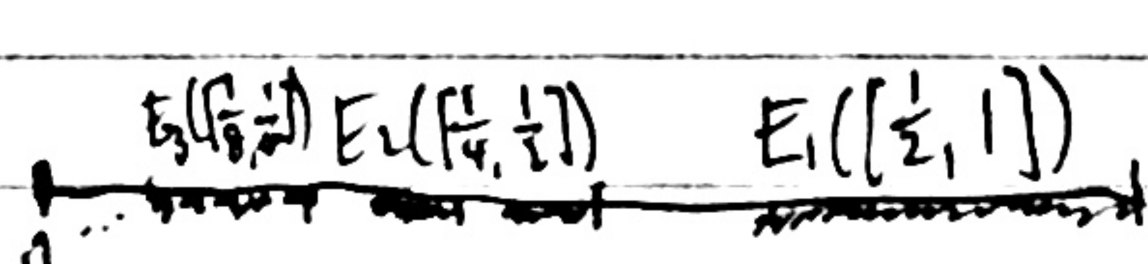


$E_3([a,b])$:



In words, $E_k([a,b])$ splits the interval $[a,b]$ into 2^{k-1} intervals of equal size, and for each subinterval, colour a fraction α of the interval starting from the right end. Coloured interval denotes the set of reals that we take in.

Now, I will construct a set E with 0 as a balanced density α point.



For each interval $[\frac{1}{2^k}, \frac{1}{2^{k+1}}]$, apply E_k to it. as shown in the diagram to the left. After constructing this interval from $[0,1]$, flip this interval over to $[-1,0]$.

I claim that 0 has density α. (both density and balanced density).

I will prove balanced density first. Consider an interval $[-w,w]$ centered at 0. Let $\frac{1}{2^k} < w < \frac{1}{2^{k+1}}$ for some $k \in \mathbb{Z}$. Then, by construction,

$$\frac{m([-w,w] \cap E)}{m([-w,w])} = \frac{m([0,w] \cap E)}{m([0,w])}$$ (by symmetry) $\Big\}$ next page because no space

Note: W lands in $E_K\left(\left[\frac{1}{2^{k-1}}, \frac{1}{2^k}\right]\right)$

So the subintervals are of length $\frac{1}{2^{k-1}}\left(\frac{1}{2^{k-1}} - \frac{1}{2^k}\right) = \frac{1}{2^{2k-1}}$

$$\frac{m([0,w]\cap E)}{m([0,w])} = \frac{m\left(\left[0,\frac{1}{2^k}\right]\cap E\right) + m\left(\left[\frac{1}{2^k}, w\right]\cap E\right)}{m([0,w])}$$

$$= \frac{\alpha\cdot\frac{1}{2^k} + \left\lfloor \frac{w-\frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \alpha \cdot \frac{1}{2^{2k-1}} + C}{m([0,w])}$$

Here, $\left\lfloor \frac{w-\frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor$ is the number of subintervals entirely contained by $\left[\frac{1}{2^k}, w\right]$

C is the remainder covered (less than $(1-\alpha)\frac{1}{2^{2k-1}}$)

$$= \frac{\alpha\left(\frac{1}{2^k} + \left\lfloor \frac{w-\frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \frac{1}{2^{2k-1}}\right) + C}{m([0,w])}$$

$$\Rightarrow \left| \frac{m([0,w]\cap E)}{m([0,w])} - \alpha \right| = \left| \frac{\alpha\left(\frac{1}{2^k} + \left\lfloor \frac{w-\frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \frac{1}{2^{2k-1}}\right) + C - \alpha w}{w} \right|$$

$$= \left| \frac{\alpha\left(\frac{1}{2^k} + \left\lfloor \frac{w-\frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \frac{1}{2^{2k-1}}\right) + C - \alpha\left(\frac{1}{2^k} + w - \frac{1}{2^k}\right)}{w} \right|$$

denotes the fractional part $\{x\} = x - \lfloor x \rfloor$

$$\le \left| \frac{\left\{ \frac{w-\frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\} \cdot \frac{1}{2^{2k-1}} + (1-\alpha)\frac{1}{2^{2k-1}}}{w} \right| \le \left| \frac{1\cdot\frac{1}{2^{2k-1}} + (1-\alpha)\frac{1}{2^{2k-1}}}{w} \right|$$

$$= \left| \frac{(2-\alpha)\frac{1}{2^{2k-1}}}{w} \right| \le \left| \frac{(2-\alpha)\frac{1}{2^{2k-1}}}{\frac{1}{2^{k-1}}} \right|$$

$$= \frac{(2-\alpha)}{2^k}$$

Hence, $\lim\limits_{Q\downarrow 0} \left| \frac{m(Q\cap E)}{m(Q)} - a \right| \le \lim\limits_{k\to\infty} \frac{2-a}{2^k} = 0$

$$\therefore \lim_{Q\downarrow 0} \left| \frac{m(Q\cap E)}{m(Q)} - a \right| = 0 \Rightarrow \boxed{\lim_{Q\downarrow 0} \frac{m(Q\cap E)}{m(Q)} = a}$$

(c) The same construction, which is measurable, works for regular density, with squares not centered.
Let $\max(Q)$ denote the maximum absolute value of the ~~integrable~~ interval containing 0, i.e. $\max([a,b]) = \max(|a|,|b|)$ provided $0 \in [a,b]$.

Then $\lim_{Q \to \{0\}} \left| \frac{m(Q \cap E)}{m(Q)} - a \right| = \lim_{\max Q \to 0} \left| \frac{m(Q \cap E)}{m(Q)} - a \right| \leq \lim_{k \to \infty} \frac{2-a}{2^k} \cdot 2 = 0$

where k is the variable s.t. $\frac{1}{2^k} < \max Q < \frac{1}{2^{k-1}}$

The multiplication by 2 is because the other side of the interval is bounded by the max side.

$\Rightarrow \boxed{\lim_{Q \to 0} \frac{m(Q \cap E)}{m(Q)} = a}$ regardless of whether Q is centered at 0 or not.

(d) Not sure what this problem is asking.
If for a specific α, then of course we can construct a measurable set with two points, one of $\delta = a$ and one of balanced $\delta = \alpha$. Just repeat the above construction at two different points.

If the problem is asking about $\forall \alpha \in [0,1]$. Then I can show that no measurable set can satisfy such conditions. Consider the function $f(x) = \frac{1}{x}$. This function preserves density everywhere except at 0. (i.e. if $x \neq 0$ has density α. Then after the transformation, $\frac{1}{x}$ has density α)

So, after applying this transformation, all points are mapped to $[-1,1]$. In particular, there are an uncountably many points in $[-1,1]$ with nonzero density and nonone.

Problem 5
6.66

I referenced a solution on math stackexchange/questions/172753 which provided the construction. I finished the rest of the proof myself.

Let Q be an enumeration of the rationals : $\{q_1, q_2, \dots\}$.
Construct the function $f: [0,1] \to \mathbb{R}$ s.t.

$$f(x) = \sum_{q_n \leq x} \frac{1}{2^n}.$$

This function is strictly increasing. Consider $x < y$. Then, by density of rationals, $\exists\, q$ s.t. $x < q < y \Rightarrow f(y) \geq \frac{1}{2^{\text{position}(q)}} + f(x) > f(x)$

This function is discontinuous at every rational input.

$$\lim_{x \to q^-} f(x) = \lim_{x \to q^-} \sum_{q_n < x} \frac{1}{2^n} \quad \text{which converges}$$

$$\neq \sum_{q_n \leq x} \frac{1}{2^n} = f(q).$$ Hence the function is not continuous at every rational number