

MATH 105 HW 7

Problem 1. f, g measurable \Rightarrow fg measurable by Exercise 28.
 6.39 f^2, g^2 integrable $\Rightarrow \int f^2, \int g^2 < \infty \Rightarrow \int f^2 + g^2 < \infty$.

Note that $(f-g)^2 \geq 0 \Rightarrow f^2 + g^2 \geq 2fg \Rightarrow \frac{f^2 + g^2}{2} \geq fg$.

Hence,

$$\int fg \leq \int \frac{f^2 + g^2}{2} = \frac{1}{2} \int f^2 + \frac{1}{2} \int g^2 < \infty \Rightarrow fg \text{ also integrable}$$

By Proposition 6.38, the set of measurable functions $f: \mathbb{R} \rightarrow \mathbb{R}$ is a vector space
 the set of integrable functions is a subspace, and integral is a linear map from
 the set of integrable functions to \mathbb{R}^n .

Define the inner product $\langle f, g \rangle = \int fg$. It satisfies positive definiteness since
 $\langle f, f \rangle = \int f^2 \geq 0$ with equality only when $f^2 = 0$ a.e. i.e. $f = 0$ a.e. (we
 consider two functions to be equivalent if $f = g$ a.e.) Also, it satisfies symmetry,
 as $\langle f, g \rangle = \int fg = \int gf = \langle g, f \rangle$. Lastly, it is bilinear, $\langle cf + h, g \rangle$
 $= \int (cf + h)g = \int cfg + hg = \int cfg + \int hg = c \int fg + \int hg = c \langle f, g \rangle + \langle h, g \rangle$

Hence, by Cauchy-Schwarz inequality $|\langle f, g \rangle|^2 \leq \langle f, f \rangle \langle g, g \rangle$

$$\Rightarrow \left(\int fg \right)^2 \leq \int f^2 \int g^2$$

$$\Rightarrow \int fg \leq \sqrt{\int f^2 \int g^2} \text{ as desired.}$$

Problem 2
6.48

$$H: [0,1] \rightarrow (0,1)$$

$$H(x+n) = H(x) + n \text{ for } n \in \mathbb{Z}$$

$$H_k(x) = \hat{H}(3^k x) \quad J(x) = \sum_{k=0}^{\infty} \frac{H_k(x)}{4^k}$$

Firstly, I will show J is continuous. We are given that H is continuous, therefore, \hat{H} is continuous since it would be continuous $\forall x \notin \mathbb{Z}$ and at $\forall x \in \mathbb{Z}$, the left and right limits are both equal to x . Hence H_k is continuous $\forall k$. $J(x)$ is the uniform limit of continuous functions, since $J(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{H_k(x)}{4^k} \Rightarrow J$ is continuous.

Next I will show J is strictly increasing

$$J(x) = \sum_{k=0}^{\infty} \frac{H_k(x)}{4^k} = \sum_{k=0}^{\infty} \frac{\hat{H}(3^k x)}{4^k} = \sum_{k=0}^{\infty} \frac{\hat{H}(\{3^k x\}) + \lfloor 3^k x \rfloor}{4^k}$$

Here, $\{x\} = x - \lfloor x \rfloor$
denotes the fractional part of x

$$= \sum_{k=0}^{\infty} \frac{H(\{3^k x\}) + \lfloor 3^k x \rfloor}{4^k}$$

Let's restrict our attention to the form $J(x) = \sum_{k=0}^{\infty} \frac{\hat{H}(3^k x)}{4^k}$.
Note $\hat{H}(x)$ is a ~~strict~~ non-decreasing function (inherited this from H)
Also, let $x < y$, then $\exists k \in \mathbb{Z}$ s.t. $y - x > \frac{1}{3^k}$.

$$\text{Then } \frac{\hat{H}(3^k y)}{4^k} = \frac{\hat{H}(3^k(y-x) + 3^k x)}{4^k} \geq \frac{\hat{H}(3^k x + 1)}{4^k} = \frac{\hat{H}(3^k x) + 1}{4^k} > \frac{\hat{H}(3^k x)}{4^k}$$

$\therefore J(y) > J(x)$ since the other terms are at least as big

$\therefore J$ is strictly increasing.

Now, let's discover some properties of J : Firstly, J satisfies $J(x+1) = J(x) + 4$.

Proof:

$$J(x+1) = \sum_{k=0}^{\infty} \frac{H(\{3^k(x+1)\}) + \lfloor 3^k(x+1) \rfloor}{4^k} = \sum_{k=0}^{\infty} \frac{H(\{3^k x + 3^k\}) + \lfloor 3^k x + 3^k \rfloor}{4^k}$$

$$= \sum_{k=0}^{\infty} \frac{H(\{3^k x\}) + \lfloor 3^k x \rfloor + 3^k}{4^k} = 4 + \sum_{k=0}^{\infty} \frac{H(\{3^k x\}) + \lfloor 3^k x \rfloor}{4^k}$$

Note $\{3^k x + 3^k\} = \{3^k x\}$ since $3^k \in \mathbb{Z}$. $= 4 + J(x)$

Hence $J(x+1) = J(x) + 4$. This means suffices to restrict our focus to the interval $[0,1]$.

Now, note that $J(x) = \sum_{k=0}^{\infty} \frac{\hat{H}(3^k x)}{4^k}$

$$J(3x) = \sum_{k=0}^{\infty} \frac{\hat{H}(3^k(3x))}{4^k} = \sum_{k=0}^{\infty} \frac{\hat{H}(3^{k+1}x)}{4^k} \Rightarrow \frac{J(3x)}{4} = \sum_{k=0}^{\infty} \frac{\hat{H}(3^{k+1}x)}{4^{k+1}} = \sum_{k=1}^{\infty} \frac{\hat{H}(3^k x)}{4^k}$$

$$\Rightarrow \boxed{J(x) = \hat{H}(x) + \frac{J(3x)}{4}}$$

Fractional part of $3x$

Assuming x belongs to constant interval $\forall k$, $J'(x) = J'(3x) \cdot \frac{3}{4}$

$$\Rightarrow \boxed{J'(x) = \frac{3}{4} J'(3x)}$$

Since $J(x+1) = J(x) + 4$, $J'(x+1) = J'(x) \Rightarrow \boxed{J'(x) = \frac{3}{4} J'(3x)}$

Now, consider the interval $[0,1]$. We ignore all rational number (since they are a measure 0 set).

Suppose $x \in [0,1]$, $x \notin \mathbb{Q}$ is such that $J'(x) = a > 0$

Then, $J'(3^k x) = \left(\frac{4}{3}\right)^k a \Rightarrow J'(\{3^k x\}) = \left(\frac{4}{3}\right)^k a$ where $\{3^k x\} \in [0,1]$ as well.

Let $S = \{x \mid J'(x) > 5\}$. Then I claim that ~~f has a derivative~~ $\forall \exists x \in [0,1]$

$x \in [0,1]$ s.t. $J'(x) = a > 0$, then $m^*(S) = 1$. This is because by the above logic, $x \notin \mathbb{Q}$

$J'(\{3^k x\}) = \left(\frac{4}{3}\right)^k a$. Hence, $\exists K$ s.t. $\forall k > K$, $\left(\frac{4}{3}\right)^k a > 5$. Also, $\{3^k x\}$ is dense in the interval $[0,1]$ since x is not a rational number. Hence, $m^*(S) = 1$.

However, this contradicts the fact that $J(0) = 0$ and $J(1) = 4$ and J is strictly increasing. Let S' be a Vitali covering of S . Then $\forall \epsilon > 0$, I will have a countable collection of ^{disjoint} intervals I_1, I_2, \dots s.t. $m\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum m(I_i) < m^*(S) + \epsilon$

However, for an interval $I_i = [a_i, b_i]$, $\frac{J(b_i) - J(x)}{b_i - x} > 5$, $\frac{J(x) - J(a_i)}{x - a_i} > 5$

$$\Rightarrow J(b_i) - J(x) > 5(b_i - x), J(x) - J(a_i) > 5(x - a_i) \Rightarrow J(b_i) - J(a_i) > 5(b_i - a_i) = 5m([a_i, b_i])$$

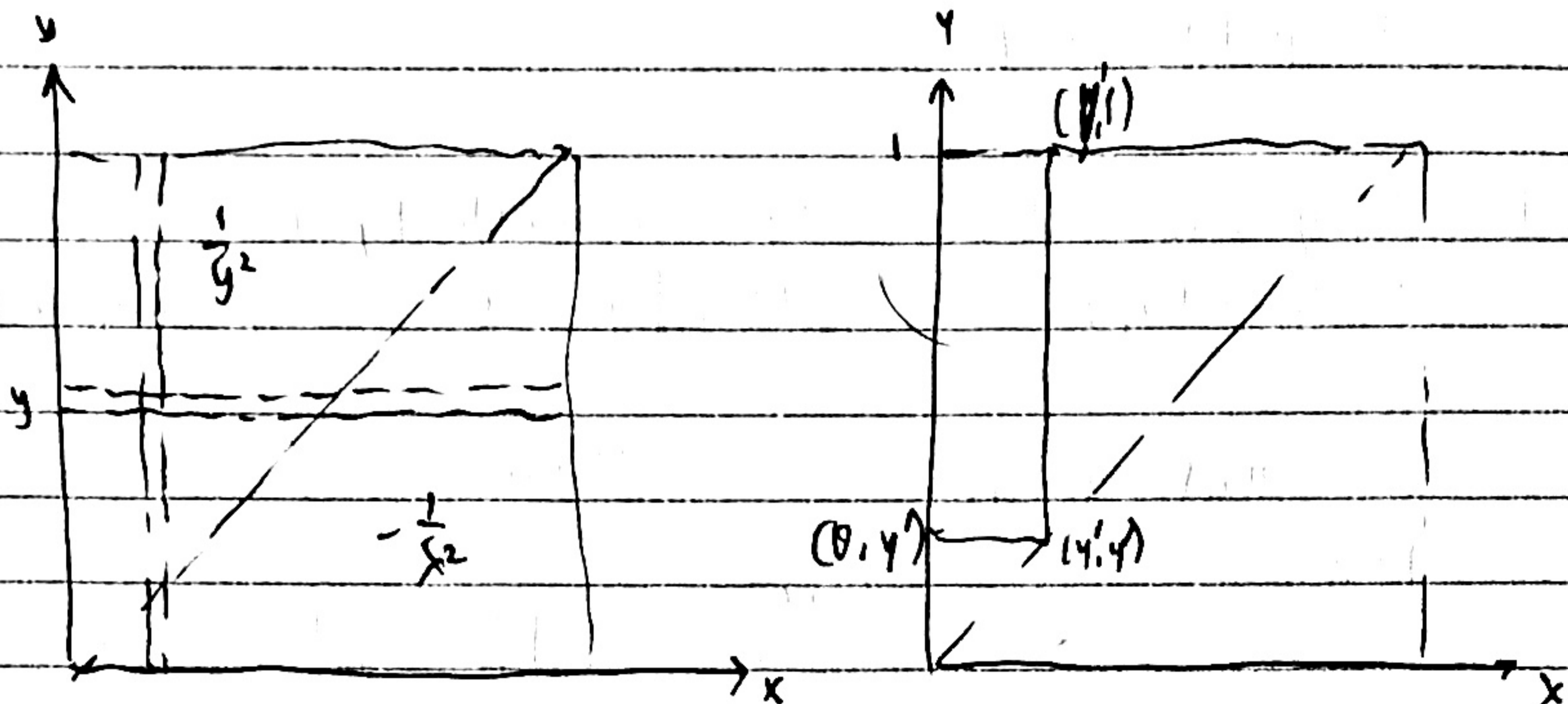
Hence $m\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum_{i=1}^{\infty} m(I_i) > 5 \sum_{i=1}^{\infty} m(I_i) < 5 \cdot J(1) - J(0) = 4$ Hence ~~par~~ (contradiction!)

Hence, the outer measure of all x on ~~any~~ intervals with $J'(x) > 0$ and x irrational must be 0. In fact, I think my proof showed that there cannot exist an irrational point where the gradient exists and > 0 and the point is on a ~~any~~ interval.

Hence $J'(x) = 0$ almost everywhere

Problem 3.
6.53

$f: \mathbb{R}^2 \rightarrow \mathbb{R}$



$$(a) \int f(x,y) dx = \int_0^y \frac{1}{y^2} dx + \int_y^1 -\frac{1}{x^2} dx = \left[\frac{1}{y^2} x \right]_0^y + \left[\frac{1}{x} \right]_y^1$$

$$= \frac{1}{y} + 1 - \frac{1}{y} = \boxed{1} \text{ finite.}$$

$$\int f(x,y) dy = \int_0^x -\frac{1}{x^2} dy + \int_x^1 \frac{1}{y^2} dy = \left[-\frac{1}{x^2} y \right]_0^x + \left[-\frac{1}{y} \right]_x^1$$

$$= -\frac{1}{x} - 1 + \frac{1}{x} = \boxed{-1} \text{ finite.}$$

~~$\iint f(x,y) dA$. For simplicity, consider the region (rectangle bounded by $(0,y)$ and $(y,1)$)~~

~~The measure of the under graph can be computed by~~

In particular, I will just show the upper \int integral does not exist.

$$\iint f dA = \int_0^1 \int_0^y \frac{1}{y^2} dx dy = \int_0^1 \left[-\frac{1}{y} \right]_0^y \frac{1}{y} dy = \left[\ln y \right]_0^1 = \infty.$$

Meanwhile, the integral of the lower triangle is similarly $-\infty$. Hence double integral does not exist.

(b) The only reason why it does not contradict is because 4.3 assumes $f: \mathbb{R}^2 \rightarrow [0, \infty)$. Here, our function is both negative and positive. If in particular, instead of $-\frac{1}{x^2}$, we have $\frac{1}{x^2}$, then all three integrals will be equal and $= \infty$.

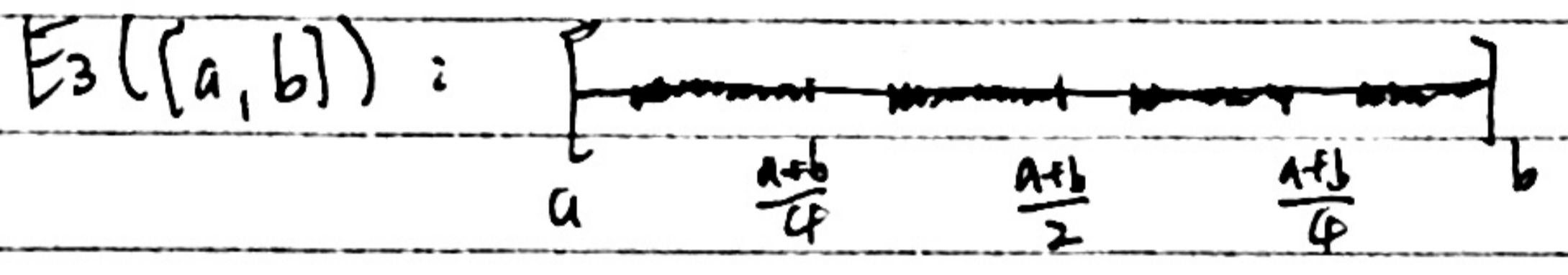
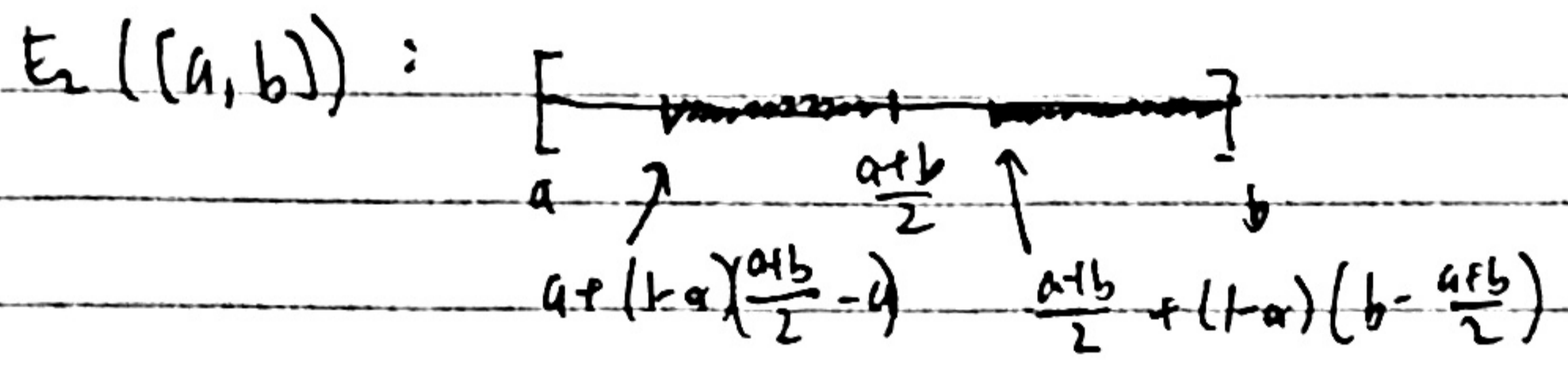
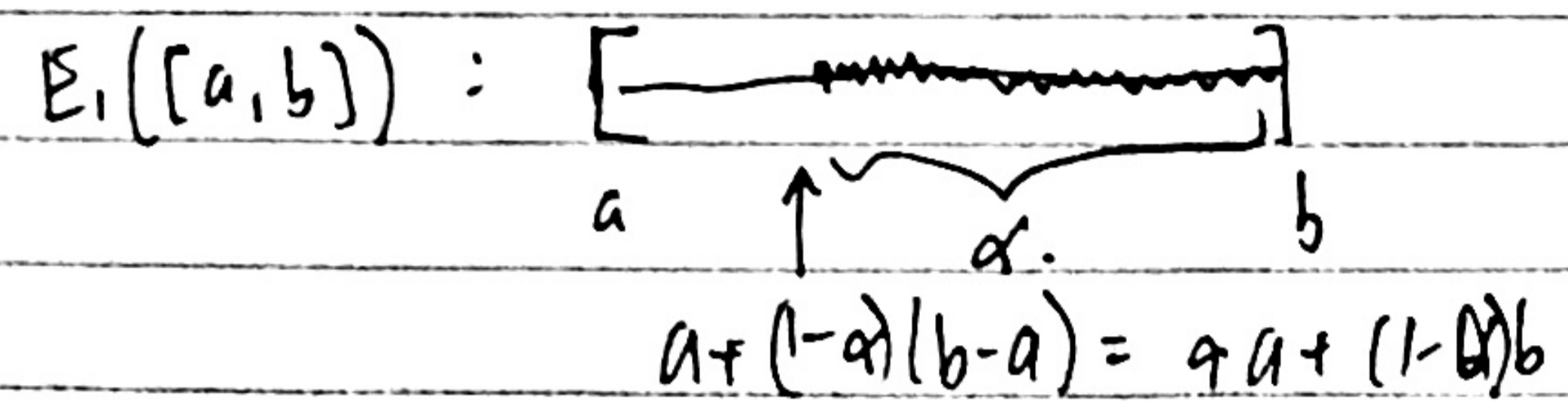
Hence, does not contradict 4.3.

Problem 4
6.58

(A) The density of E at p is defined as $d(p, E) = \lim_{\alpha \downarrow p} \frac{m(E \cap \alpha)}{m(\alpha)}$ where the cubes need not be centered at p . In particular, if p is a density point of E , then p automatically satisfies the condition for being a balanced density point (a more restrictive definition). Hence, almost every point of E is a balanced density point.

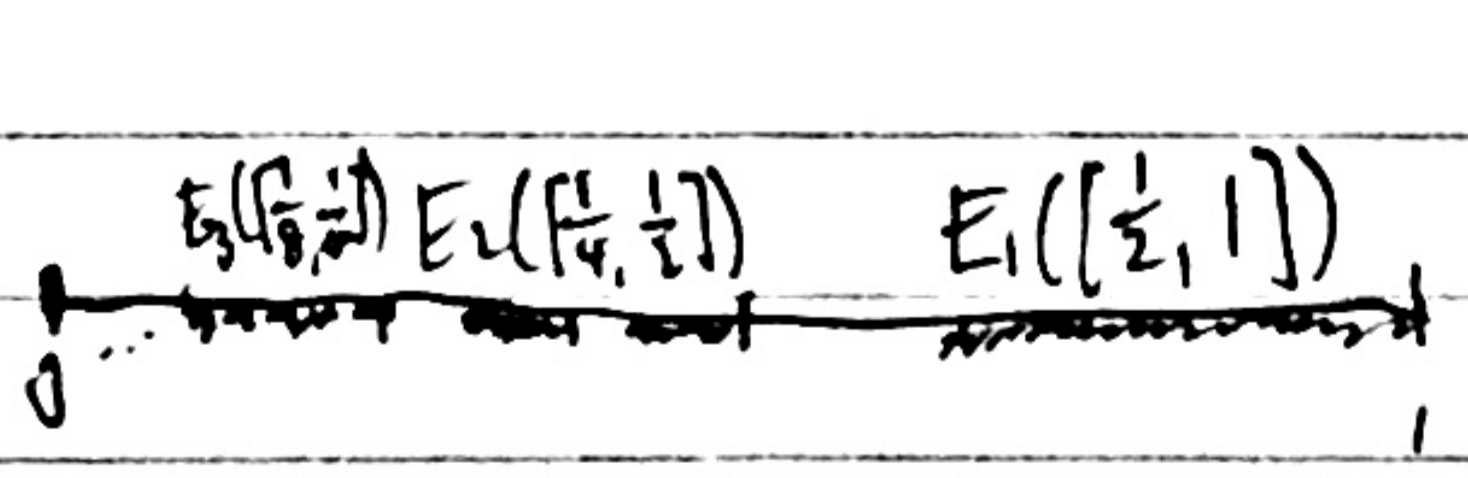
Let $\alpha \in [0, 1]$

(b) Consider a function $E_k([a, b])$ that performs the following operation on the interval $[a, b]$:



In words, $E_k([a, b])$ splits the interval $[a, b]$ into 2^{k-1} intervals of equal size, and for each subinterval, colour a fraction α of the interval starting from the right end. Coloured interval denotes the set of reals that we take m .

Now, I will construct a set E with 0 as a balanced density point.



For each interval $[\frac{1}{2^k}, \frac{1}{2^{k+1}}]$, apply E_k to it. as shown in the diagram to the left. After constructing this interval from $[0, 1]$, flip this interval over to $[-1, 0]$.

I claim that 0 has density α . (both density and balanced density).

I will prove balanced density first. Consider an interval $[-w, w]$ centered at 0 . Let $\frac{1}{2^k} < w < \frac{1}{2^{k+1}}$ for some $k \in \mathbb{Z}$. then, by construction,

$$\frac{m([-w, w] \cap E)}{m([-w, w])} = \frac{m([0, w] \cap E)}{m([0, w])} \quad (\text{by symmetry})$$

next page because no space

Note: w lands in $E_k \left(\left[\frac{1}{2^{k-1}}, \frac{1}{2^k} \right] \right)$

So the subintervals are of length $\frac{1}{2^{k-1}} \left(\frac{1}{2^{k-1}} - \frac{1}{2^k} \right) = \frac{1}{2^{2k-1}}$

$$\frac{m([0, w] \cap E)}{m([0, w])} = \frac{m\left([0, \frac{1}{2^k}] \cap E\right) + m\left([\frac{1}{2^k}, w] \cap E\right)}{m([0, w])}$$

$$= \frac{\alpha \cdot \frac{1}{2^k} + \left\lfloor \frac{w - \frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \alpha \cdot \frac{1}{2^{2k-1}} + c}{m([0, w])}$$

Here, $\left\lfloor \frac{w - \frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor$ is the number of subintervals entirely contained by $[\frac{1}{2^k}, w]$

$$= \frac{\alpha \left(\frac{1}{2^k} + \left\lfloor \frac{w - \frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \frac{1}{2^{2k-1}} \right) + c}{m([0, w])}$$

c is the remainder covered (less than $\frac{1}{2^{2k}}$)

$$\Rightarrow \left| \frac{m([0, w] \cap E)}{m([0, w])} - \alpha \right| = \left| \frac{\alpha \left(\frac{1}{2^k} + \left\lfloor \frac{w - \frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \frac{1}{2^{2k-1}} \right) + c - \alpha w}{w} \right|$$

$$= \left| \frac{\alpha \left(\frac{1}{2^k} + \left\lfloor \frac{w - \frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\rfloor \cdot \frac{1}{2^{2k-1}} \right) + c - \alpha \left(\frac{1}{2^k} + w - \frac{1}{2^k} \right)}{w} \right|$$

denotes the fractional part
 $\{x\} = x - \lfloor x \rfloor$

$$\leq \left| \frac{\left\{ \frac{w - \frac{1}{2^k}}{\frac{1}{2^{2k-1}}} \right\} \cdot \frac{1}{2^{2k-1}} + (1 - \alpha) \frac{1}{2^{2k-1}}}{w} \right| \leq \left| \frac{1 \cdot \frac{1}{2^{2k-1}} + (1 - \alpha) \frac{1}{2^{2k-1}}}{w} \right|$$

$$\leq \left| \frac{(2 - \alpha) \frac{1}{2^{2k-1}}}{w} \right| \leq \left| \frac{(2 - \alpha) \frac{1}{2^{2k-1}}}{\frac{1}{2^{k-1}}} \right|$$

$$\text{Hence, } \lim_{Q \downarrow 0} \left| \frac{m(Q \cap E)}{m(Q)} - \alpha \right| \leq \lim_{k \rightarrow \infty} \frac{2 - \alpha}{2^k} = 0$$

$$\therefore \lim_{Q \downarrow 0} \left| \frac{m(Q \cap E)}{m(Q)} - \alpha \right| = 0 \Rightarrow \boxed{\lim_{Q \downarrow 0} \frac{m(Q \cap E)}{m(Q)} = \alpha}$$

which is measurable

(c) The same construction works for regular density, with squares not centered
 Let $\max(Q)$ denote the maximum absolute value of the ~~interval~~ interval
 containing Q , i.e. $\max([a,b]) = \max(|a|, |b|)$ provided $0 \in [a,b]$.

$$\lim_{Q \rightarrow 0} \left| \frac{m(Q \cap E)}{m(Q)} - a \right| = \lim_{\max Q \rightarrow 0} \left| \frac{m(Q \cap E)}{m(Q)} - a \right| \leq \lim_{k \rightarrow \infty} \frac{2^{-k}}{2^k} = 0$$

where k is the variable s.t.

$$\frac{1}{2^k} < \max Q < \frac{1}{2^{k-1}}$$

$$\lim_{Q \rightarrow 0} \frac{m(Q \cap E)}{m(Q)} = a$$

irrespective of whether Q is
 centered at 0 or not

The multiplication by 2 is
 because the other side of the
 interval is bounded by the
 max side.

(d) Not sure what this problem is asking.

If for a specific α , then of course we can construct
 a measurable set with the points, one of $\delta = \alpha$ and
 one of balanced $\delta = \alpha$. Just repeat the above construction at two different points.

If the problem is asking about $\forall \alpha \in [0,1]$. Then I can show that no measurable
 set can satisfy such conditions. Consider the function $f(x) = \frac{1}{x}$. This function
 preserves density everywhere except at 0. (i.e. if x^0 has density α . Then after
 the transformation, $\frac{1}{x}$ has density α)

So, after applying this transformation, all points are mapped to $[-1,1]$. In particular,
 there are uncountably many points in $[-1,1]$ with nonzero density
 and none.

Problem 5
6.66

I referenced a solution on math stackexchange/questions/172753 which provided the construction. I finished the rest of the proof myself.

Let Q be an enumeration of the rationals: $\{q_1, q_2, \dots\}$.

Construct the function $f: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$f(x) = \sum_{q_n \leq x} \frac{1}{2^n}.$$

This function is strictly increasing. Consider $x < y$. Then, by density of rationals,

$$\exists q \text{ s.t. } x < q < y \Rightarrow f(y) \geq \frac{1}{2^{\text{position}(q)}} + f(x) > f(x)$$

This function is discontinuous at every rational input.

$$\lim_{x \rightarrow q_i^-} f(x) = \lim_{x \rightarrow q_i^-} \sum_{q_n \leq x} \frac{1}{2^n} \text{ which converges}$$

$$\neq \sum_{q_n \leq q_i} \frac{1}{2^n} = f(q_i). \text{ Hence the function is not continuous at every rational number.}$$