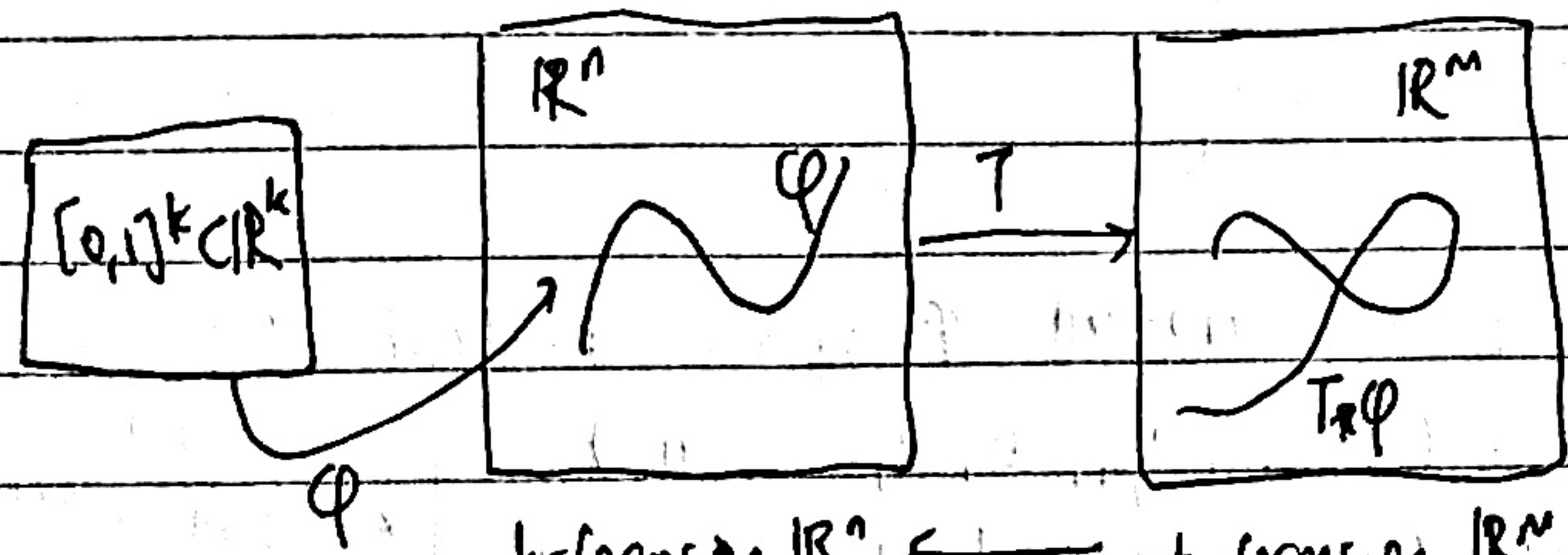
04/07/2022

MATH 105 PRESENTATION NOTES

Pushforward and Pullback

$$\underset{\varphi : [0,1]^k \to \mathbb{R}^n}{\text{k-cells in } \mathbb{R}^n} \xrightarrow[\text{pushforward}]{T_*} \underset{\varphi : [0,1]^k \to \mathbb{R}^m}{\text{k-cells in } \mathbb{R}^m}$$



$T : \mathbb{R}^n \to \mathbb{R}^m$

$$\text{k-forms on } \mathbb{R}^n \xleftarrow[\text{pullback}]{T^*} \text{k-forms on } \mathbb{R}^m$$

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth transformation, then it induces a natural transformation of k-cells in $\mathbb{R}^n$ to k-cells in $\mathbb{R}^m$ via composition i.e. $T_* : \underbrace{\varphi}_{[0,1]^k \to \mathbb{R}^n} \to \underbrace{T \circ \varphi}_{[0,1]^k \to \mathbb{R}^m}$

Call $T_*\varphi$ the pushforward of T.

Say that the k-cell φ in $\mathbb{R}^n$ gets pushed forward to become k-cell $T_*\varphi$ on $\mathbb{R}^m$.

Dual to pushforward is the pullback i.e. α is k-form on $\mathbb{R}^m$ (recall k-form in $\mathbb{R}^m$: k-cell in $\mathbb{R}^m \to \mathbb{R}$), then k-form $T^*\alpha$ takes k-cell in $\mathbb{R}^n$ φ and sends it to a real number $\alpha(T\varphi)$

i.e. $\boxed{T^*(\alpha(\varphi)) = \alpha(T_*(\varphi))}$

Properties of Pullback

(a) If α is a k-form on $\mathbb{R}^m$, then $T^*\alpha$ is a k-form on $\mathbb{R}^n$.

In particular, $T^*(dz_{i_1, i_2, \dots, i_k}) = T^*(dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_k})$
$= dT_{i_1} \wedge dT_{i_2} \wedge \dots \wedge dT_{i_k}$.

(b) Preserves wedge products. $T^*(\alpha \wedge \beta) = T^*\alpha \wedge T^*\beta$

(c) Commutes with exterior derivative: $dT^* = T^*d$.

(d) Commutes with integral: $\int_{T \circ \varphi} \alpha = \int_\varphi T^*\alpha$.

Do I need to prove these?

The General Stokes Formula

Definition: A k-chain is a linear combination of k-cells i.e. $\Phi = \sum_{j=1}^{N} a_j \varphi_j$.

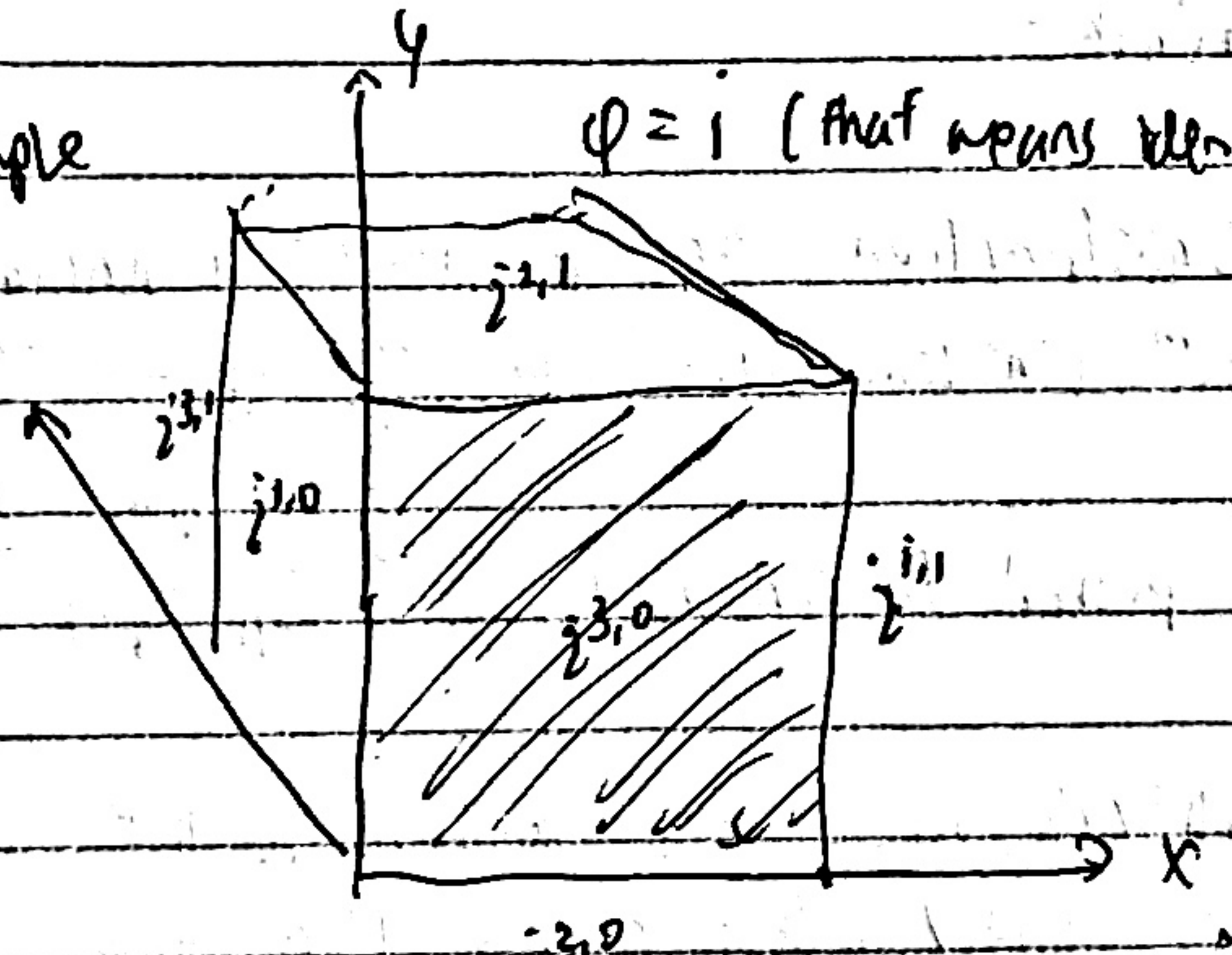
Then integral of a k-chain is just the integral over the separate k-cells

$$\int_{\Phi} \omega = \int_{\sum_{j=1}^{N} a_j \varphi_j} \omega = \sum_{j=1}^{N} a_j \int_{\varphi_j} \omega$$

Definition: The boundary of a $(k+1)$-cell φ is the k-chain $\partial\varphi = \sum_{j=1}^{k+1} (-1)^{j+1} (\varphi \circ i^{j,1} - \varphi \circ i^{j,0})$

where $i^{j,0}(u_1, \dots, u_k) = (u_1, \dots, u_{j-1}, 0, u_j, \dots, u_k)$
$i^{j,1}(u_1, \dots, u_k) = (u_1, \dots, u_{j-1}, 1, u_j, \dots, u_k)$] k-cells in $\mathbb{R}^{k+1}$

Example $\varphi = i$ (that means identity, makes $[0,1]^{k+1} \to \mathbb{R}^{k+1}$)



$i^{3,0} = [0,1]^2 \to [a, b, 0]$ $0 \le a, b \le 1$

Given a $(k+1)$-cell, $i^{j,1}$, $i^{j,0}$ chooses the j^{th} face and tracks where it is mapped to via $\varphi \circ i^{j,1}$, $\varphi \circ i^{j,0}$ therefore called the boundary.

Definition: define the j^{th} dipole of φ as $\delta^j \varphi = \varphi \circ i^{j,1} - \varphi \circ i^{j,0}$ (the j^{th} term in the sum)

i.e. $\partial \varphi = \sum_{j=1}^{k+1} (-1)^{j+1} \delta^j \varphi$

Stoke's Formula for Cubes

Let ω be a $(n-1)$-form in $\mathbb{R}^n$ (ie $k = n-1$) and $i : [0,1]^n \to \mathbb{R}^n$ be the identity then n-cell.

Then $\int_i d\omega = \int_{\partial i} \omega$

Proof: $\omega = \sum_{i=1}^{n} f_i(x)\, dx_1 \wedge dx_2 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$.

Then $d\omega = \sum_{i=1}^{n} d f_i(x) \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$

$= \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i} dx_i \wedge dx_1 \wedge \dots \widehat{dx_i} \wedge dx_n = \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i} (-1)^{i+1} dx_1 \wedge \dots \wedge dx_n$

$= \left[\sum_{i=1}^{n} (-1)^{i+1} \frac{\partial f_i(x)}{\partial x_i} \right] dx_1 \wedge \dots \wedge dx_n.$

Hence, $\int_i d\omega = \int_i \left(\sum_{i=1}^n (-1)^{i-1} \frac{\partial f_i}{\partial x_i}\right) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$

$= \sum_{i=1}^n (-1)^{i-1} \int_{[0,1]^n} \frac{\partial f_i}{\partial x_i} |J| dx_1 \, dx_2 \dots dx_n$

Jacobian is the identity matrix

$x_i \to x_i$

$J = I$

On the other side,

$\int_{\partial i} \omega = \int_{\sum_{j=1}^n (-1)^{j+1} \delta^j_i} \omega = \sum_{j=1}^n \cancel{\int_{(-1)^{j+1}}} \sum_{j=1}^n (-1)^{j+1} \int_{\delta^j_i} \omega$

$\delta^j_i = i \circ i^{j,1} - i \circ i^{j,0} = i^{j,1} - i^{j,0}$ (recall $i^{j,1}, i^{j,0} : \mathbb{R}^{n-1} \to \mathbb{R}^n$ adding ~~one~~ coordinate at jth position by 0 or 1)

~~where~~

$\int_{\delta^j_i} \omega = \int_{\delta^j_i} \sum_{\ell=1}^n f_\ell(x) \, dx_1 \wedge \dots \wedge \widehat{dx_\ell} \wedge \dots \wedge dx_n$

$\mathbb{R}^{n-1}$

$\boxed{u_1, u_2, \dots u_{n-1}}$

$= \cancel{\int_{\delta^j_i}} \cancel{\int_{[0,1]^{n-1}}} \int_{i^{j,1} - i^{j,0}} \sum_{\ell=1}^n f_\ell(x) \, dx_1 \wedge \dots \wedge \widehat{dx_\ell} \wedge \dots \wedge dx_n$

$= \int_{[0,1]^{n-1}} \left(f_j(i^{j,1}(u)) - f_j(i^{j,0}(u)) \right) du_1 \, du_2 \dots du_{n-1}$

$\cancel{= \int_{[0,1]^{n-1}} \int_0^1 \frac{\partial f_j}{\partial x_j} dx_j}$

$$\begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \dots & \frac{\partial x_1}{\partial u_{n-1}} \\ & & & \\ & & & \end{bmatrix}$$

If x_j is inside $\frac{\partial x_j}{\partial u_i} = 0$

$= \int_{[0,1]^{n-1}} \Big(f_j((u_1, u_2, \dots, u_{j-1}, 1, u_j, \dots, u_n))$

$- f_j(u_1, u_2, \dots, u_{j-1}, 0, u_j, \dots, u_n) \Big) du_1 \, du_2 \dots du_{n-1}$

$\Rightarrow x_j$ cannot be inside

Fundamental Theorem of Calculus.

$= \int_{[0,1]^{n-1}} \left(\int_0^1 \frac{\partial f_j}{\partial x_j} dx_j \right) du_1 \, du_2 \dots du_{n-1}$

Fubini's Theorem swapping the order of integration

$= \int_{[0,1]^n} \frac{\partial f_j}{\partial x_j} dx_1 \, dx_2 \dots dx_j \, d \dots dx_n$ (just renaming the variables).

Hence, $\int_{\partial i} \omega = \sum_{j=1}^n (-1)^{j+1} \int_{\delta^j_i} \omega = \sum_{j=1}^n (-1)^{j+1} \int_{[0,1]^n} \frac{\partial f_j}{\partial x_j} dx_1 \dots dx_n. \left(= \int_i d\omega \right)$

$\therefore$ Stoke's ~~Theorem~~ Formula for ω be proved here ~~(what about $k \neq n$?)~~ ??

ω is $(n-1)$ form on $\mathbb{R}^m$
$d\omega$ is n form on $\mathbb{R}^m$

Stoke's Formula for General cells..

If ω is an $(n-1)$-form in $\mathbb{R}^m$ and ϕ is an n-cell in $\mathbb{R}^m$, then $\int_\phi d\omega = \int_{\partial\phi} \omega$

$$\int_\phi d\omega = \int_{\phi \circ i} d\omega = \int_i \underbrace{\phi^* d\omega}_{n \text{ form on } \mathbb{R}^m} = \int_i d \overbrace{\phi^* \omega}^{n-1 \text{ form on } \mathbb{R}^m} = \int_{\partial i} \phi^* \omega = \int_{\phi_* \partial i} \omega$$

$$= \int_{\partial\phi} \omega$$

$\phi: [0,1]^n \to \mathbb{R}^m$ $\quad \phi: \mathbb{R}^n \to \mathbb{R}^m$ $\quad [0,1]^n \to \mathbb{R}^n$ $\quad$ n-forms on $\mathbb{R}^m$ to n-forms on $\mathbb{R}^n$ $\quad i: [0,1]^n \to \mathbb{R}^n$ itself $[0,1]^n$

What's the big picture behind this?

1st equality: identity inclusion map, so $\phi = \phi \circ i$

2nd equality: Follows from 4.3 (d) $\int_{T \circ \varphi} \alpha = \int_\varphi T^* \alpha$ (definition of pull back)

$\int_{T \circ \varphi} \alpha = \int_{T_* \varphi} \alpha = \alpha(T_* \varphi) = T^*(\alpha(\varphi)) = \int_\varphi T^* \alpha$

3rd equality: Follows from 4.3 (c) commutative property of exterior derivative and pull back operator $\phi^* d\omega = d\phi^* \omega$ (?)

4th equality: Stoke's Formula for cubes $\int_i d\phi^* \omega = \int_{\partial i} \phi^* \omega$ valid since $\phi^* \omega$ is $n-1$ form on $\mathbb{R}^n$

5th equality: Duality equation $T^*(\alpha)(\phi) = \alpha\, T_*(\phi)$

$\Rightarrow \int_\phi T^* \alpha = \int_{T_*(\phi)} \alpha$

Final equality follows from the fact that $\phi_*(\partial i) = \phi_*\left(\sum_{j=1}^{k+1} (-1)^{j+1} (i \circ i^{j,1} - i \circ i^{j,0})\right)$

$$= \phi_*\left(\sum_{j=1}^{k+1} (-1)^{j+1} (i^{j,1} - i^{j,0})\right)$$

$$= \phi \circ \left(\sum_{j=1}^{k+1} (-1)^{j+1} (i^{j,1} - i^{j,0})\right)$$

$$= \sum_{j=1}^{k+1} (-1)^{j+1} (\phi \circ i^{j,1} - \phi \circ i^{j,0}) = \partial\phi$$

Stoke's Formula for Manifolds

Let ω be a $(m-1)$-form in $\mathbb{R}^n$ and $M \subset \mathbb{R}^n$ divides into m-cells diffeomorphic to I^m and its boundary divides into $(m-1)$ cells diffeomorphic to $[0,1]^{m-1}$, then $\int_M d\omega = \int_{\partial M} \omega$.

General Spirit

$$\int_{\partial M} \omega = \int_M d\omega$$

integrate over boundary of manifold can be reduced to integrating over the entire manifold

Vector Calculus.

Fundamental Theorem of Calculus

$$\int_M d\omega = \int_{\partial M} \omega$$

① $M = [a,b] \subset \mathbb{R}$ $\quad \omega = f : \mathbb{R}^2 \to \mathbb{R}$

$$\int_{\partial M} \omega = \int_{\{a,b\}} f = f(b) - f(a)$$

$$\int_M d\omega = \int_M df = \int_{[a,b]} f_x \, dx \quad \therefore \int_a^b f_x \, dx = f(b) - f(a).$$

② M = path in $\mathbb{R}^2$ from p to q. $\quad \omega = f : \mathbb{R}^2 \to \mathbb{R}$ smooth.

$\partial M = p, q.$

$$\int_{\partial M} \omega = \int_{\{p,q\}} f = f(q) - f(p)$$

$$\int_M d\omega = \int_M df = \int_M f_x \, dx + f_y \, dy \quad \therefore \int_{\substack{\text{path} \\ \text{from } p \text{ to } q}} f_x \, dx + f_y \, dy = f(q) - f(p)$$

which corresponds with what we know path independent

③ Green's.

$\omega = f\,dx + g\,dy.$ $\quad M = D$ be a region $\quad C$ be its boundary

$$\int_{\partial M} \omega = \int_C f\,dx + g\,dy$$

$$\int_M d\omega = \int_D df \wedge dx + dg \wedge dy = \int_D \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy$$

$$= \int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

$$= \int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy.$$

④ Divergence $F = (f, g, h)$ be smooth vector field. $M = V$. $\partial M = S$

consider $\omega = f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$

$$\int_{\partial M} \omega = \int_S f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$$

$$\int_M d\omega = \int_V df \wedge dy \wedge dz + dg \wedge dz \wedge dx + dh \wedge dx \wedge dy$$

$$\int_V \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \right) dx \wedge dy \wedge dz$$

$$= \int_V \operatorname{div} F \, dx\,dy\,dz.$$

Stoke's Curl Theorem

$\omega = f\,dx + g\,dy + h\,dz$, $\quad M = S$ surface, $\partial M = C$ curve

$$\int_{\partial M} \omega = \int_C \omega = \int_C f\,dx + g\,dy + h\,dz$$

$$\int_M d\omega = \int_M df \wedge dx + dg \wedge dy + dh \wedge dz$$

$$= \int_S \frac{\partial f}{\partial y}\,dy \wedge dx + \frac{\partial f}{\partial z}\,dz \wedge dx$$

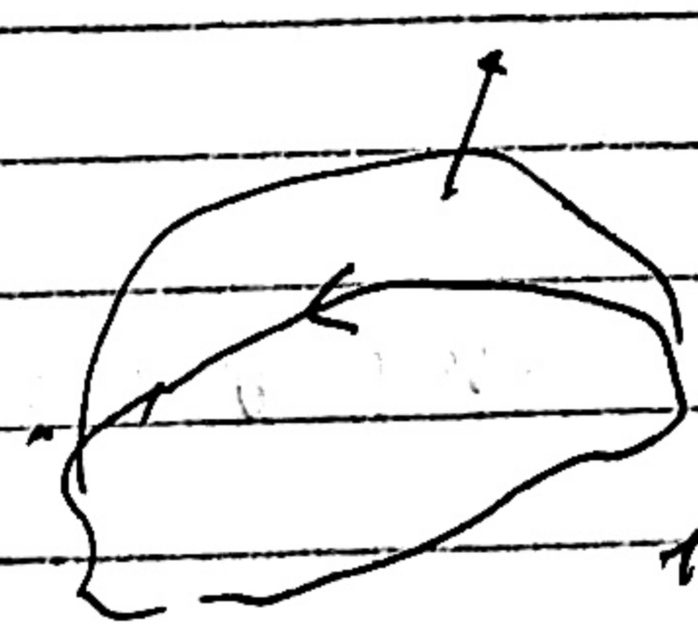
$$+ \frac{\partial g}{\partial x}\,dx \wedge dy + \frac{\partial g}{\partial z}\,dz \wedge dy$$

$$+ \frac{\partial h}{\partial x}\,dx \wedge dz + \frac{\partial h}{\partial y}\,dy \wedge dz$$

$$= \int_S \cancel{\left(\frac{\partial h}{\partial y}\right) dy \wedge dz +} \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz$$

$$+ \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx$$

$$+ \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$



the line integral of vector field over a loop is equal to flux of its curl through the enclosed surface