1. Rational zero Theorem (Ross §2).
$\mathbb{Z}$.

Def: an integer coeff polynomial in x is

$$f(x) = c_n \cdot x^n + c_{n-1} \cdot x^{n-1} + \cdots + c_1 \cdot x + c_0$$

$c_n, \cdots, c_0 \in \mathbb{Z}$.
$c_n \neq 0$

$\mathbb{Z}$- coeff eq. is : $f(x) = 0$

one can ask: when does an $\mathbb{Z}$- coeff eq has roots in $\mathbb{Q}$.

fact: a degree n polynomial has n roots in $\mathbb{C}$.

i.e. $\exists\ z_1, \cdots, z_n$ in $\mathbb{C}$, such that.

$$f(x) = c_n (x - z_1) \cdots (x - z_n).$$

(it is possible that some of the z_i coincide.)

Thm: If a rational number r satisfies the eq.

$$c_n \cdot x^n + \cdots + c_1 x^1 + c_0 = 0. \quad , \quad \text{with } c_i \in \mathbb{Z},\ c_n \neq 0.$$

and $r = \frac{c}{d}$ (where c, d are co-prime integers). Then.

c divides c_0, and. d divides c_n.

Ex: (1). $5x + 3 = 0$, $r = \frac{3}{5}$. $c = 3$, $d = 5$.

$c_1 = 5$, $c_0 = 3$. yes. $3 | 3$, $5 | 5$.

Pf: Plug in $x = \frac{c}{d}$, to eq.

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0$$

multiply both sides by d^n, we get

$$C_n \cdot c^n + C_{n-1} \cdot c^{n-1} \cdot d + \cdots + C_1 \cdot c \cdot d^{n-1} + C_0 \cdot d^n = 0.$$

(1) $\because$ $C_n \cdot c^n = -\left(C_{n-1} \cdot c^{n-1} \cdot d + \cdots + C_1 \cdot c \cdot d^{n-1} + C_0 \cdot d^n\right)$

$= -d\left(C_{n-1} \cdot c^{n-1} + \cdots + C_0 \cdot d^{n-1}\right)$

$\therefore$ d divides $C_n \cdot c^n$.

since d and c are coprime, d does not divide c^n

$\therefore$ d has to divide C_n. $\boxed{d \mid 0 \ ✓}$

(2) $C_0 \cdot d^n = -\left(C_n \cdot c^n + C_{n-1} \cdot c^{n-1} \cdot d + \cdots + C_1 \cdot c \cdot d^{n-1}\right)$

$= -c \cdot \left(C_n \cdot c^{n-1} + C_{n-1} \cdot c^{n-2} \cdot d + \cdots + C_1 \cdot d^{n-1}\right).$

by similar reasoning, $c \mid C_0$. $\square$.

Using this rational zero thm: we can answer questions

claim: (Ex 4).

$\sqrt[3]{6}$ is not rational number.

$\Leftrightarrow$ $x^3 - 6 = 0$ does not have rational roots.

Pf: The only possible rational sol'n $r = \frac{c}{d}$, needs

$c \mid 6$, $d \mid 1$, $\therefore$ take $d = 1$, $c = \pm 1, \pm 2, \pm 3, \pm 6$.

one can test all of them, they don't solve the Eq

$\therefore$ there is no sol'n in $\mathbb{Q}$. $\square$.

- $\mathbb{R}$

Historical construction of $\mathbb{R}$ from $\mathbb{Q}$:

(1) Dedekind cut: (Q: if $\sqrt{2} \notin \mathbb{Q}$, how to "save the

info" of $\sqrt{2}$?).

$C_{\sqrt{2}} = \{ r \in \mathbb{Q} \mid r < \sqrt{2} \}$. a subset.

moral: for each $x \in \mathbb{R}$, consider $C_x = \{ r \in \mathbb{Q} \mid r < x \}$.

one can define addition, multiplication on these subsets. C_x.

(2). <u>Sequence in $\mathbb{Q}$</u> i.e. to use a seq of rational numbers to "approximate" a real number.

e.g. $\sqrt{2}$ can be approx by

$1,\ 1.4,\ 1.41,\ 1.414,\ \cdots\cdots$.

<u>problem here</u>: ① given any real number, how do you get such a seq ?

② how to tell if 2 different sequences approx the same real number.

(e.g.

$1 \leftarrow 1.1,\ 1.01,\ 1.001,\ \cdots$

$1 \leftarrow 0.9,\ 0.99,\ 0.999,\ \cdots$

or $1 \leftarrow 1,\ 1,\ 1,\ 1,\ \cdots$)

the 3 sequences all have the <u>same limit</u> (what is a limit ?)

• Given the existence of $\mathbb{R}$. we have properties (<u>axioms</u>) of $\mathbb{R}$.

• <u>completeness of $\mathbb{R}$</u> (axiom):

Given any subset $E \subset \mathbb{R}$, bounded above, there exist a unique. $r \in \mathbb{R}$

① r is an <u>upper bound</u> of E

② for any other <u>upper bound</u> α, we have $r \le \alpha$.

<u>bounded above</u>: $\exists\, a \in \mathbb{R}$, s.t. for any $x \in E$, we have $x \le a$.

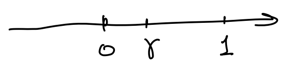
r is called the least upper bound, of E, $r = \sup E$.

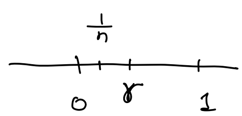
(i.e. $\sup(E)$ is well ~~ed~~ defined for subset E that's bounded above).

Ex: $\sup\left([0,1]\right) = 1$. , $\sup\left((0,1)\right) = 1$.

$$\sup\left(\{r \in \mathbb{Q} \mid r^2 < 2\}\right) = \sqrt{2}$$

Cor: (Archaemedian property): For any $r \in \mathbb{R}$, $r > 0$, $\exists\ n \in \mathbb{N}$, such that $n \cdot r > 1$. $\Leftrightarrow r > \frac{1}{n}$.





$+\infty, -\infty$: notations.

- With these symbol introduced, we can say
  $\sup(\mathbb{N}) = +\infty \iff \mathbb{N}$ is not bounded above.
- $+\infty, -\infty$ are not real numbers. They have partly the operations that $\mathbb{R}$ has. i.e.
  $3 \cdot (+\infty) = +\infty$, $(-3) \cdot (+\infty) = -\infty$,

but $(+\infty) + (-\infty) \neq$ NAN. $(0) \cdot (+\infty) =$ not defined.

Sequences and Limits:

- a sequence of real numbers is: $a_0, a_1, a_2, \cdots\cdots$, denoted as $(a_n)_{n=0}^{\infty}$ or shortened (a_n).

note, use $(\cdot\cdot)$, not $\{\cdot\cdot\}$

- we only care about the "eventual behavior" of a seq.

- <u>Def</u>: A seq (a_n) <u>converge</u> to $a \in \mathbb{R}$, if $\forall \underline{\varepsilon} > 0$, $\exists N \in \mathbb{N}$, such that, $\forall n \geq N$, $|a_n - a| < \varepsilon$.