Rudin 12

12. Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of R^2 into R^3 by

$$f_1(s,t) = (b + a\cos s)\cos t$$
$$f_2(s,t) = (b + a\cos s)\sin t$$
$$f_3(s,t) = a\sin s.$$

Describe the range K of $\mathbf{f}$. (It is a certain compact subset of R^3.)

(*a*) Show that there are exactly 4 points $\mathbf{p} \in K$ such that

$$(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = \mathbf{0}.$$

Find these points.

(*b*) Determine the set of all $\mathbf{q} \in K$ such that

$$(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = \mathbf{0}.$$

(*c*) Show that one of the points $\mathbf{p}$ found in part (*a*) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points").

I describe K geometrically. Here is the cross-sections of X in 3 planes:

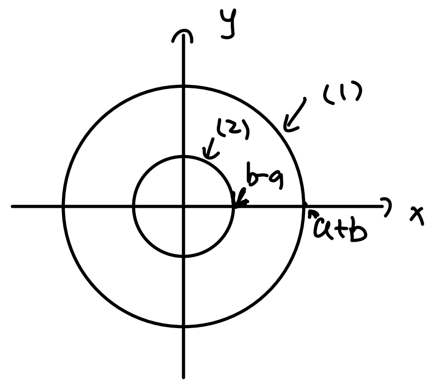
Let $f(s,t) = (x, y, z)$.

1) $z = 0$ (x-y plane)

$z=0 \Rightarrow s = n\pi,\ n = 0, 1, 2, \dots$

$\Rightarrow$ (1) $\begin{cases} x = (b+a)\cos t \\ y = (b+a)\sin t \end{cases}$

$\Rightarrow x^2+y^2 = (b+a)^2$



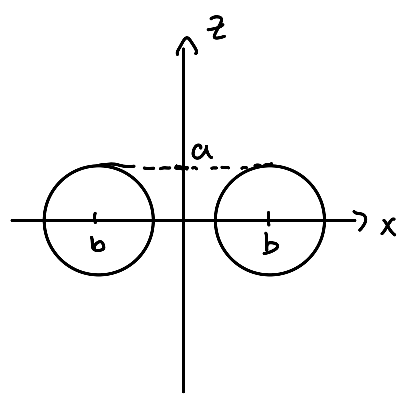
(2) $\begin{cases} x = (b-a)\cos t \\ y = (b-a)\sin t \end{cases}$

$\Rightarrow x^2+y^2 = (b-a)^2$

2) $y=0$ (x-z plane)

(1) $(x-b)^2 + z^2 = a^2$

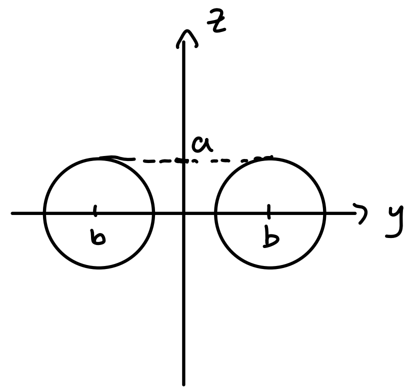
(2) $(x+b)^2 + z^2 = a^2$



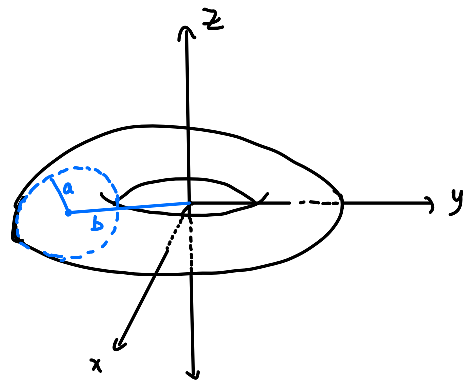
3) $x=0$ ($y-z$ plane)

(1) $(y-b)^2+z^2=a^2$

$(y+b)^2+z^2=a^2$



Therefore, K is a "donut" shape (or a tube with its ends connected).
b is distance from the center of the "donut hole" to the center of the tube,
a is the radius of the tube.



(a) First, compute ∇f_1,

$$\nabla f_1 = \langle -a\sin s\cos t,\ -(b+a\cos s)\sin t \rangle$$

For $\nabla f_1(s,t)=0$, we need

1) $\sin s=0 \Rightarrow s=n\pi,\ n=0,1,2,\dots$

or

$\cos t=0 \Rightarrow t=n\frac{\pi}{2},\ n=1,2,\dots$

and

2) $\sin t=0 \Rightarrow t=n\pi,\ n=0,1,2,\dots$

$\Rightarrow s=n\pi,\ t=n\pi,\ n=0,1,\dots$

$\Rightarrow s=0,\pi$

$t=0,\pi$

Therefore, mapping $(s,t)\mapsto(x,y,z)$,

$(0,0) \mapsto (b+a,0,0)$

$(\pi,0) \mapsto (b-a,0,0)$

$(0,\pi) \mapsto (-b-a,0,0)$

$(\pi,\pi) \mapsto (a-b,0,0)$

(b) $\nabla f_3 = \langle a\cos(s), 0\rangle$

For $\nabla f_3(s,t) = 0$, we need

$\cos(s)=0 \Rightarrow s = n\frac{\pi}{2} = \frac{\pi}{2}, \frac{3\pi}{2}$

Therefore, mapping $(\frac{\pi}{2},t)$, $(\frac{3\pi}{2},t)$, we have

$x = b\cos t$

$y = b\sin t$

$z = a, -a$

$\Rightarrow x^2+y^2 = b^2$

So the set we want is 2 circles centered at $(0,0,a)$ and $(0,0,-a)$ with radius b.

(c) $f_1(s,t) = (b+a\cos(s))\cos(t)$, this is maximized everywhere when $\cos(s) = \cos(t) = 1$, which is achieved at $s=t=0$ so the point $(a+b,0,0)$ is a maximum. f_1 is minimized when $\cos(s)=1$, $\cos(t)=-1$, this is achieved at $s=0$, $t=\pi$, so the point $(-a-b,0,0)$ is a minimum.

To see why $(a-b,0,0)$ and $(b-a,0,0)$ are neither, consider

$f_1(\pi,t) = (b-a)\cos(t) \Rightarrow t=0$ maximizes this

$f_1(s,0) = b+a\cos(s) \Rightarrow s=\pi$ minimizes this

Therefore,

$$f_1(\pi,t) < f(\pi,0) < f(s,0) \quad \text{for } s,t \text{ close}$$

So $(b-a,0,0)$, the point in K that $(\pi,0)$ maps to is neither a max or a min. Similarly, $(a-b,0,0)$ is also neither.

$f_3(s,t) = a\sin(s)$, which is maximized when $s=\frac{\pi}{2}$ and minimized when $s=\frac{3\pi}{2}$. Therefore, f_3 is maximized on the top circle centered at $(0,0,a)$ with radius b and minimized on the bottom circle centered at $(0,0,-a)$ with radius b.

Rudin 13

13. Suppose $\mathbf{f}$ is a differentiable mapping of R^1 into R^3 such that $|\mathbf{f}(t)| = 1$ for every t. Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$.

Interpret this result geometrically.

Let $f(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$, then since $|f(t)| = 1$,

$$f_1^2(t) + f_2^2(t) + f_3^2(t) = 1$$

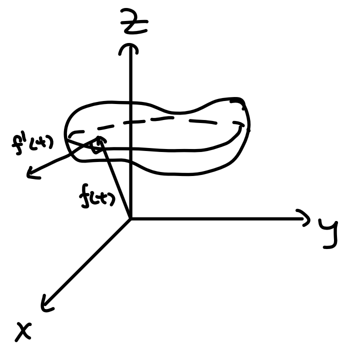
$$\Rightarrow \frac{d}{dt} f_1^2(t) + f_2^2(t) + f_3^2(t) = 0$$

$$\Rightarrow \cancel{2} f_1(t) f_1'(t) + \cancel{2} f_2(t) f_2'(t) + \cancel{2} f_3(t) f_3'(t) = 0$$

$$\Rightarrow \langle f_1(t), f_2(t), f_3(t) \rangle \cdot \langle f_1'(t), f_2'(t), f_3'(t) \rangle = 0$$

$$\Rightarrow f(t) \cdot f'(t) = 0$$

Geometrically, the vector $f(t)$, is perpendicular to f's tangent vector at t.



<u>Rudin 19</u>

19. Show that the system of equations

$$3x + y - z + u^2 = 0 \qquad (1)$$
$$x - y + 2z + u = 0 \qquad (2)$$
$$2x + 2y - 3z + 2u = 0 \qquad (3)$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

First, we notice that

$$(1) - (2) - (3) : \; u^2 - 3u = 0 \Rightarrow u = 0, 3$$

So the system is solved if and only if $u = 0$ or 3.

Let $f = (f_1, f_2, f_3)$, where

$$f_1(x,y,z,u) = 3x + y - z + u^2$$
$$f_2(x,y,z,u) = x - y + 2z + u$$
$$f_3(x,y,z,u) = 2x + 2y - 3z + 2u$$

Reparametrize $f(x,y,z,u)$ as $f(h,k)$, where $h \in \mathbb{R}$, $k \in \mathbb{R}^3$.

Then we can try to apply the implicit function theorem and show that if $f(h_0,k_0) = 0$, for a given h, k is uniquely determined when h, k are sufficiently close to h_0, k_0, and thus the system can be solved for entries of k written in terms of h.

For ImFT to apply, we need to show that $B = \left[\frac{\partial f_i(h,k)}{\partial k_j}\right]$ is invertible. $(\det(B) = 0)$

We calculate the determinant 4 times.

1) $h = z$, $k = (x, y, u)$

$$\det(B) = \det\begin{pmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix} = 8u - 12$$

$\overset{u=0}{=} -12 \neq 0$

$\overset{u=3}{=} 12 \neq 0$

2) $h = y, \quad k = (x, z, u)$

$$\det(B_2) = \det\begin{pmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} = 21 - 14u \quad \begin{matrix} \overset{u=0}{=} 21 \neq 0 \\ \underset{u=3}{=} -31 \neq 0 \end{matrix}$$

3) $h = x, \quad k = (y, z, u)$

$$\det(B_3) = \det\begin{pmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix} = 3 - 2u \quad \begin{matrix} \overset{u=0}{=} 3 \neq 0 \\ \underset{u=3}{=} -3 \neq 0 \end{matrix}$$

4) $h = u, \quad k = (x, y, z)$

$$\det(B_4) = \det\begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix} = 0$$

ImFT applies to cases 1, 2, 3, but not 4. To further show that x, y, z can not be written in terms of u to solve the system, notice

$$B_4 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -u^2 \\ -u \\ -2u \end{bmatrix}$$

But since B_4 is not invertible, this equation has no solution.

Pugh 14

14. An $n \times n$ matrix is **diagonalizable** if there is a change of basis in which it becomes diagonal.

(a) Is the set of diagonalizable matrices open in $\mathcal{M}_n$?

(b) Closed?

(c) Dense?

For this problem, the metric we use is 1-norm-induced, namely,

$$d(A,B) = \|A-B\|_1,$$

where $\|\cdot\|_1$ is the 1-norm of the colum with maximum 1-norm (from HW8).

From Discord discussion with Samuel, since $\mathcal{M}_n$ is a vector space, norm equivalence applies. I choose the 1-norm, although I believe, say, the max-norm, is also fine. Further more, I prove results in $\mathcal{M}_2$, but I think the results should generalize to n well.

Lastly let $S = \{M \in \mathcal{M}_2 : M \text{ is diagonalizable}\}$

(a) Overview: W.T.S. that $\forall\, \varepsilon > 0$, the ε-ball centered around a diagonalizable matrix is not contained in S.

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & \frac{\varepsilon}{2} \\ 0 & 1 \end{bmatrix}, \quad \varepsilon > 0$$

Then we know:

(1) $A \in S$,

(2) $d(A,W) = \left\| \begin{matrix} 0 & -\frac{\varepsilon}{2} \\ 0 & 0 \end{matrix} \right\|_1 = \frac{\varepsilon}{2}$

$\Rightarrow W \in B_\varepsilon(A)$

(3) W is diagonalizable

proof of (3):

First find the eigenvalues of W:

$\det(W-\lambda I) = (1-\lambda)^2$

$\Rightarrow \lambda = 1$ (w/ multiplicity 2)

Then find the dimension of the eigenspace of W corresponding to $\lambda=1$.

$W - 1\cdot I = \begin{bmatrix} 0 & \frac{\varepsilon}{2} \\ 0 & 0 \end{bmatrix} \Rightarrow \dim(E_1(A)) = 1 \neq 2$

$\Rightarrow W$ is not diagonalizable

From above and since ε is arbitrary, $B_\varepsilon(A) \not\subset S \ \forall \ \varepsilon > 0$,

thus S is not open.

(b) Overview: W.T.S. : $\exists$ a limit point of S that is not in S.

Let

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1+\frac{\varepsilon}{2} & 1 \\ 0 & 1 \end{bmatrix}$$

We know

(1) B is not diagonalizable (from (a)(3))

(2) V is diagonalizable (V has 2 distinct eigenvalues) $\Rightarrow V \in S$

(3) $d(B, V) = \frac{\varepsilon}{2} \Rightarrow V \in B_\varepsilon(B)$

$\Rightarrow B_\varepsilon(B) \cap S \neq \emptyset$

$\Rightarrow B$ is a limit point of S since ε is arbitrary

Since S does not contain its limit point B, S is not closed.

(c) Overview: W.T.S. $\exists\, M \in \mathcal{M}_n$ s.t. $M \notin S$ and M is not a limit point of S.

Thus we show that $\overline{S} \neq \mathcal{M}_n$.

The following construction follows from the Math StackExchange answer by Adren.

Let

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The characteristic polynomial of R is λ^2+1, and $\lambda^2+1=0$ has no real solution (its discriminant is negative).

Since R has no real eigenvalues, it is not diagonalizable.

Then, assume that $\exists\, (D_n)$ a sequence of matrices in S s.t.

$$D_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}, \text{ where } \begin{matrix} a_n \to 0, & b_n \to -1, \\ c_n \to 1, & d_n \to 0 \end{matrix}$$

Overall, $D_n \to R$. Since $D_n \in S$, its characteristic polynomial

$$(a_n - \lambda)(d_n - \lambda) - b_n c_n$$

has discriminant

$$a_n^2 + d_n^2 - 2a_n d_n + 4 b_n c_n \geq 0$$

This implies

$$\lim_{n\to\infty} a_n^2 + d_n^2 - 2a_n d_n + 4 b_n c_n \geq 0$$

But,

$$\lim_{n\to\infty} a_n^2 + d_n^2 - 2a_n d_n + 4 b_n c_n = -4 < 0$$

Therefore, no such (D_n) exists. So R is not a limit point of S.

Since $R \notin S$ also, $\overline{S} \neq \mathcal{M}_n$, so S is not dense in $\mathcal{M}_n$.

Pugh 24

24. Show that all second partial derivatives of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,\ 0) \end{cases}$$

exist everywhere, but the mixed second partials are unequal at the origin, $\partial^2 f(0,0)/\partial x \partial y \neq \partial^2 f(0,0)/\partial y \partial x$.

At $(x,y) \neq (0,0)$,

$$D_x f(x,y) = \frac{x^4 y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

$$D_y f(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

$$D_x(D_x f)(x,y) = \frac{12xy^5 - 4x^3y^3}{(x^2+y^2)^3}$$

$$D_y(D_y f)(x,y) = \frac{4x^3y^3 - 12x^5y}{(x^2+y^2)^3} = D_x(D_y f)(x,y)$$

At $(0,0)$,

$$D_x f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$D_y f(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

$$D_x(D_x f)(0,0) = \lim_{h\to 0} \frac{D_x f(h,0) - D_x f(0,0)}{h} = 0$$

$$D_y(D_y f)(0,0) = \lim_{h\to 0} \frac{D_y f(0,h) - D_y f(0,0)}{h} = 0$$

$$D_x(D_y f)(0,0) = \lim_{h \to 0} \frac{D_y f(h,0) - D_y f(0,0)}{h} = 1$$

$$D_y(D_x f)(0,0) = \lim_{h \to 0} \frac{D_x f(0,h) - D_x f(0,0)}{h} = -1 \neq D_x(D_y f)(0,0)$$