Claim 1: $d\Omega_1 = 0$

$$\Omega_1 = |X|^{-2} x_1 dx_2 - |X|^{-2} x_2 dx_1$$

$$\Rightarrow d\Omega_1 = d(|X|^{-2} x_1) \wedge dx_2 - d(|X|^{-2} x_2) \wedge dx_1$$

$$= d\left(\frac{x_1}{x_1^2+x_2^2}\right) \wedge dx_2 - d\left(\frac{x_2}{x_1^2+x_2^2}\right) \wedge dx_1$$

$$= \frac{\partial}{\partial x_1} \frac{x_1}{x_1^2+x_2^2} dx_1 \wedge dx_2 - \frac{\partial}{\partial x_2} \frac{x_2}{x_1^2+x_2^2} dx_2 \wedge dx_1$$

$$= \left(\frac{x_1^2+x_2^2-2x_1^2}{(x_1^2+x_2^2)^2}\right) dx_1 \wedge dx_2 + \left(\frac{x_1^2+x_2^2-2x_2^2}{(x_1^2+x_2^2)^2}\right) dx_1 \wedge dx_2$$

$$= \frac{x_1^2+x_2^2-2x_1^2+x_1^2+x_2^2-2x_2^2}{(x_1^2+x_2^2)^2} dx_1 \wedge dx_2$$

$$= 0$$

Claim 2: $d\Omega_2 = 0$

$$d\Omega_2 = d(|X|^{-3} x_1 dx_2 \wedge dx_3) - d(|X|^{-3} x_2 dx_1 \wedge dx_3) + d(|X|^{-3} x_3 dx_1 \wedge dx_2)$$

$$= d(|X|^{-3} x_1 dx_{(2,3)}) - d(|X|^{-3} x_2 dx_{(1,3)}) + d(|X|^{-3} x_3 dx_{(1,2)})$$

$$= \frac{\partial}{\partial x_1} \frac{x_1}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} dx_{(1,2,3)} - \frac{\partial}{\partial x_1} \frac{x_2}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} dx_{(2,1,3)}$$

$$+ \frac{\partial}{\partial x_1} \frac{x_3}{(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}} dx_{(3,1,2)}$$

$$= \frac{3(x_1^2+x_2^2+x_3^2)^{\frac{3}{2}} - (x_1^2+x_2^2+x_3^2)^{\frac{3}{2}}(3)}{(x_1^2+x_2^2+x_3^2)^3} dx_{(1,2,3)}$$

$$= 0$$

Claim 3: Generally,

$$\Omega_{n-1} = |X|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$$
$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n}{2}}} \, dx_{(1,2,\dots,i-1,i+1,\dots,n)}$$

And $d\Omega_{n-1} = 0$.

$$d\Omega_{n-1} = \sum_{i=1}^{n} d\left(\frac{x_i}{\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n}{2}}}\right) (-1)^{i-1} \, dx_{(1,2,\dots,i-1,i+1,\dots,n)}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{x_i}{\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n}{2}}}\right) (-1)^{i-1} (-1)^{i-1} \, dx_{(1,2,\dots,n)}$$

$$= \sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n}{2}} - n\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n}{2}-1} x_i^2}{\left(\sum_{j=1}^{n} x_j^2\right)^{n}} \, dx_{(1,2,\dots,n)}$$

$$= |x|^{-n} \left[n\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n}{2}} - n\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n}{2}} \right] dx_{(1,2,\dots,n)}$$

$$= 0$$

S^2 parametrization 1

$$\frac{\partial \gamma_1}{\partial s} = \pi \cos(\pi s)\cos(2\pi t) \qquad \frac{\partial \gamma_2}{\partial s} = \pi \cos(\pi s)\sin(2\pi t)$$

$$\frac{\partial \gamma_1}{\partial t} = -2\pi \sin(\pi s)\cos(2\pi t) \qquad \frac{\partial \gamma_2}{\partial t} = 2\pi \sin(\pi s)\cos(2\pi t)$$

$$\frac{\partial \gamma_3}{\partial s} = -\pi \sin(\pi s), \quad \frac{\partial \gamma_3}{\partial t} = 0$$

$$\frac{\partial \gamma(1,2)}{\partial(s,t)} = \pi^2 \sin(2\pi s)$$

$$\frac{\partial \gamma(2,3)}{\partial(s,t)} = 2\pi^2 \sin^2(\pi s)\cos(2\pi t)$$

$$\frac{\partial \gamma(1,3)}{\partial(s,t)} = -2\pi^2 \sin^2(\pi s)\sin(2\pi t)$$

$$x_1^2 + x_2^2 + x_3^2 = \sin^2(\pi s)\cos^2(2\pi t) + \sin^2(\pi s)\sin^2(2\pi t) + \cos^2(\pi s)$$
$$= 1$$

$$\int_\gamma \Omega_2 = \int_{[0,1]^2} \sin(\pi s)\cos(2\pi t)\,(2\pi^2 \sin^2 \pi s)\cos(2\pi t)\, ds\,dt$$

$$- \int_{[0,1]^2} \sin(\pi s)\sin(2\pi t)\,(-2\pi^2 \sin^2(\pi s)\sin(2\pi t))\, ds\,dt$$

$$+ \int_{[0,1]^2} \cos(\pi s)\,\pi^2 \sin(2\pi s)\, ds\,dt$$

$$= \pi^2 \int_{[0,1]^2} \Big[2\sin^3(\pi s)\cos^2(2\pi t) + 2\sin^3(\pi s)\sin^2(2\pi t) + \sin(2\pi s)\cos(\pi s)\Big]\, ds\,dt$$

$$= \pi^2 \int_0^1 \int_0^1 \sin(\pi s)\, ds\,dt$$

$$= 4\pi$$

S^2 parametrization 2

$$\frac{\partial \gamma_1}{\partial a} = \frac{2-2a^2+2b^2}{(1+a^2+b^2)^2} \qquad \frac{\partial \gamma_2}{\partial a} = \frac{-4ab}{(1+a^2+b^2)^2}$$

$$\frac{\partial \gamma_1}{\partial b} = \frac{-4ab}{(1+a^2+b^2)^2} \qquad \frac{\partial \gamma_2}{\partial b} = \frac{2+2a^2-2b^2}{(1+a^2+b^2)^2}$$

$$\frac{\partial \gamma_3}{\partial a} = \frac{4a}{(1+a^2+b^2)^2} , \quad \frac{\partial \gamma_3}{\partial b} = \frac{4b}{(1+a^2+b^2)^2}$$

$$\frac{\partial \gamma_{(1,2)}}{\partial (a,b)} = \frac{4-4a^4-4b^4-8a^2b^2}{(1+a^2+b^2)^4}$$

$$\frac{\partial \gamma_{(1,3)}}{\partial (a,b)} = \frac{8b-8a^2b+8b^3}{(1+a^2+b^2)^4}$$

$$\frac{\partial \gamma_{(2,3)}}{\partial (a,b)} = \frac{-8a-8a^3-8ab^2}{(1+a^2+b^2)^4}$$

$$\int_\gamma \Omega_2 = \int_{\mathbb{R}^2} \left[\frac{2a}{1+a^2+b^2} \frac{\partial \gamma_{(2,3)}}{\partial (a,b)} - \frac{2b}{1+a^2+b^2} \frac{\partial \gamma_{(1,3)}}{\partial (a,b)} + \frac{-1+a^2+b^2}{1+a^2+b^2} \frac{\partial \gamma_{(1,2)}}{\partial (a,b)} \right) da\, db$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-4(1+a^2+b^2)^3}{(1+a^2+b^2)^5} \, da\, db$$

$$= -4\pi$$