Let $\omega = f(x,y,z)\, dx \wedge dy + g(x,y,z)\, dx \wedge dz + h(x,y,z)\, dy \wedge dz$

be a 2-form on $\mathbb{R}^3$.

As in lecture, define

$$\rho : (x,y,z,t) \mapsto (xt, yt, zt)$$

$$N : \omega \mapsto \int_0^1 \tilde{\omega}, \ \tilde{\omega} \text{ is the part of } \omega \text{ with } t\,dt$$

And let

$$f_t = f(xt,yt,zt), \quad g_t = g(xt,yt,zt), \quad h_t = (xt,yt,zt)$$

First, we have

$$\begin{aligned}
\rho^*\omega &= f_t\, d(xt) \wedge d(yt) + g_t\, d(xt) \wedge d(zt) + h_t\, d(yt) \wedge d(zt) \\
&= f_t\, (x\,dt + t\,dx) \wedge (y\,dt + t\,dy) \\
&\quad + g_t\, (x\,dt + t\,dx) \wedge (z\,dt + t\,dz) \\
&\quad + h_t\, (y\,dt + t\,dy) \wedge (z\,dt + t\,dz) \\
&= f_t \left[(t\,dt \wedge (x\,dy - y\,dx)) + t^2 dx \wedge dy \right] \\
&\quad + g_t \left[(t\,dt \wedge (x\,dy - y\,dx) + t^2 dx \wedge dz \right] \\
&\quad + h_t \left[(t\,dt \wedge (y\,dz - z\,dy) + t^2 dy \wedge dz \right]
\end{aligned}$$

Note:

① $d\rho^*\omega = \rho^* d\omega = 0$

② $$\begin{aligned}
d\rho^*\omega = \ & d\{ f_t \left[(t\,dt \wedge (x\,dy - y\,dx)) + t^2 dx \wedge dy \right] \} \\
& + d\{ g_t \left[(t\,dt \wedge (x\,dy - y\,dx) + t^2 dx \wedge dz \right] \} \\
& + d\{ h_t \left[(t\,dt \wedge (y\,dz - z\,dy) + t^2 dy \wedge dz \right] \} = 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & d f_t \left[(t\,dt \wedge (x\,dy - y\,dx)) \right] + d\left[g_t \left[(t\,dt \wedge (x\,dy - y\,dx) \right] \right. \\
& + d\left[h_t \left[(t\,dt \wedge (y\,dz - z\,dy) \right] \right. \\
& = -\{ df_t\, t^2 dx \wedge dy + dg_t\, t^2 dx \wedge dz + dh_t\, t^2 dy \wedge dz \}
\end{aligned}$$

Then, we have

$$N\rho^*\omega = \int_0^1 f_t\, t\,dt \wedge (x\,dy - y\,dx)$$
$$+ \int_0^1 g_t\, t\,dt \wedge (x\,dz - z\,dx)$$
$$+ \int_0^1 h_t\, t\,dt \wedge (y\,dz - z\,dy)$$

And,

$$dN\rho^*\omega = d\left[\int_0^1 f_t\, t\,dt\right] \wedge (x\,dy - y\,dx) + \left[\int_0^1 f_t\, t\,dt\right] \wedge d(x\,dy - y\,dx)$$
$$+ d\left[\int_0^1 g_t\, t\,dt\right] \wedge (x\,dz - z\,dx) + \left[\int_0^1 g_t\, t\,dt\right] \wedge d(x\,dz - z\,dx)$$
$$+ d\left[\int_0^1 h_t\, t\,dt\right] \wedge (y\,dz - z\,dy) + \left[\int_0^1 h_t\, t\,dt\right] \wedge d(y\,dz - z\,dy)$$

$$= \int_0^1 \Big[df_t\, t\,dt \wedge (x\,dy - y\,dx) + dg_t\, t\,dt \wedge (x\,dz - z\,dx)$$
$$+ dh_t\, t\,dt \wedge (y\,dz - z\,dy) \Big]$$
$$+ \left[\int_0^1 f_t\, t\,dt\right] \wedge d(x\,dy - y\,dx)$$
$$+ \left[\int_0^1 g_t\, t\,dt\right] \wedge d(x\,dz - z\,dx)$$
$$+ \left[\int_0^1 h_t\, t\,dt\right] \wedge d(y\,dz - z\,dy)$$

$$= -\int_0^1 \left[df_t\, t^2 dx \wedge dy + dg_t\, t^2 dx \wedge dz + dh_t\, t^2 dy \wedge dz \right]$$
$$+ \left[\int_0^1 f_t\, t\,dt\right] \wedge d(x\,dy - y\,dx)$$
$$+ \left[\int_0^1 g_t\, t\,dt\right] \wedge d(x\,dz - z\,dx)$$
$$+ \left[\int_0^1 h_t\, t\,dt\right] \wedge d(y\,dz - z\,dy)$$

$$= -\int_0^1 \frac{\partial}{\partial t}\Big[f_t\, t^2\, dx \wedge dy + (g_t\, t^2\, dx \wedge dz + h_t\, t^2\, dy \wedge dz \Big] \wedge dt$$
$$+ \left[\int_0^1 f_t\, t\,dt\right] \wedge d(x\,dy - y\,dx)$$
$$+ \left[\int_0^1 g_t\, t\,dt\right] \wedge d(x\,dz - z\,dx)$$
$$+ \left[\int_0^1 h_t\, t\,dt\right] \wedge d(y\,dz - z\,dy)$$

$$= -\left[f(x,y,z)\, dx \wedge dy + g(x,y,z)\, dx \wedge dz + h(x,y,z)\, dy \wedge dz \right]$$

$$\left. \begin{aligned} &+ \int_0^1 f_t\, t\, dt \wedge 2\, dx \wedge dy \\ &+ \int_0^1 g_t\, t\, dt \wedge 2\, dx \wedge dz \\ &+ \int_0^1 h_t\, t\, dt \wedge 2\, dy \wedge dz \end{aligned} \right\}$$

not sure how to simplify this

(Idea: need to write $f_t\, t\, dt$ as the exterior derivative of some 1-form)