Lemma 0:

By finite sub-additivity,

$$m^*(E \cup F) \leq m^*(E) + m^*(F)$$

So it suffices to show

$$m^*(E \cup F) \geq m^*(E) + m^*(F)$$

First, $\forall \varepsilon > 0$, $\exists \{B_j\}_{j \in J}$ open box covering $(E \cup F)$ s.t.

$$\sum_{j \in J} |B_j| \leq m^*(E \cup F) + \varepsilon$$

Since E and F are separated, we can divide up $\{B_j\}$ into 2 disjoint sets, $\{B_j\}_{j \in J_E}$, $\{B_j\}_{j \in J_F}$, s.t. they each cover E, F, respectively. Then we have

$$\sum_{j \in J} |B_j| = \sum_{j \in J_E} |B_j| + \sum_{j \in J_F} |B_j|$$

And since

$$m^*(E) \leq \sum_{j \in J_E} |B_j|, \quad m^*(F) \leq \sum_{j \in J_F} |B_j|,$$

we have

$$m^*(E) + m^*(F) \leq \sum_{j \in J_E} |B_j| + \sum_{j \in J_F} |B_j| = \sum_{j \in J} |B_j| \leq m^*(E \cup F) + \varepsilon$$

Since ε is arbitrary,

$$m^*(E) + m^*(F) \leq m^*(E \cup F)$$

Lemma 1:

Since $\forall\, U \supset A$, $m^*(U) \geq m^*(A)$, we have

$$\inf\{m^*(U),\ U \supset A,\ U \text{ open}\} \geq m^*(A)$$

Then it suffices to show

$$\inf\{m^*(U),\ U \supset A,\ U \text{ open}\} \leq m^*(A)$$

$\forall\, \varepsilon > 0$, $\exists\ B := \{B_j\}$ open box covering of A s.t.

$$\sum |B_j| \leq m^*(A) + \varepsilon$$

By Corollary 7.2.7 of Tao, $m^*(B_j) = |B_j|\ \forall j$, so

$$\sum |B_j| = \sum m^*(B_j) \geq m^*(B) \qquad \text{(sub-additivity)}$$

$$\geq \inf\{m^*(U) : U \supset A,\ U \text{ open}\} \quad (B \supset A,\ B \text{ open})$$

$$\Rightarrow \inf\{m^*(U) : U \supset A,\ U \text{ open}\} \leq m^*(A) + \varepsilon$$

$$\Rightarrow \inf\{m^*(U) : U \supset A,\ U \text{ open}\} \leq m^*(A) \text{ since } \varepsilon \text{ is arbitrary}$$

Lemma 2:

Since E_i is measurable $\forall i$, $\forall \varepsilon > 0$ $\exists$ open set $U_i \supset E_i$ s.t.

$$m^*(U_i \setminus E_i) < \frac{\varepsilon}{2^i}$$

Now, let $W = \bigcup_{i=1}^{\infty} U_i$ and $W_i = U_i \setminus E_i$, $E = \bigcup_{i=1}^{\infty} E_i$.

Then if $x \in W \setminus E$, then $\forall i$, $x \notin E_i$. But since $x \in W \setminus E$, $x \in W$, meaning that for some i, $x \in U_i$, so $x \in \bigcup_{i=1}^{\infty} W_i$.

Since x is arbitrary, $\bigcup_{i=1}^{\infty} W_i \supset W \setminus E$. Then,

$$\begin{aligned} m^*(W \setminus E) &\le m^*\left(\bigcup_{i=1}^{\infty} W_i\right) && \text{(monotonicity)} \\ &\le \sum_{i=1}^{\infty} m^*(W_i) && \text{(countable sub-additivity)} \\ &\le \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \\ &= \varepsilon \end{aligned}$$

Since U_i is open, W is open. Therefore, $E = \bigcup_{i=1}^{\infty} E_i$ is measurable.

Lemma 3 :

To prove this lemma, we prove the following 2 claims :

- Claim 1 : A can be written as a countable union of bounded closed subsets.

  proof : Let $B_\varepsilon(x)$ be a closed ball of radius ε centered at x, $x \in \mathbb{R}^n$. Then, we can write A as

  $$A = \bigcup_{i=1}^{\infty} (B_i(0) \cap A)$$

  Since $B_i(0)$ and A are both closed, $(B_i(0) \cap A)$ is closed.

- Claim 2 : Any bounded closed (thus compact) subset $A \subset \mathbb{R}^n$ is measurable.

  proof :

  Since A is bounded, $\exists$ open set $U \supset A$ s.t., by Lemma 1,

  $$m^*(U) < m^*(A) + \varepsilon$$

  Since U open and A closed, $U \setminus A$ is open. Then, by Lemma 7.4.10 of Tao, $U \setminus A$ can be written as a countable union of open boxes $\bigcup_{i=1}^{\infty} B_i$.

?

Lemma 4:

Since E is measurable, $\forall \varepsilon > 0$, $\exists$ open set $U_n \supset E$ s.t.,

$$m^*(U_n \setminus E) < \frac{\varepsilon}{n}, \; n \in \mathbb{N}$$

Since U_n is open, U_n^c is closed, thus measurable by Lemma 3.

Let $Q = \bigcup_{n=1}^{\infty} U_n^c$, then, by Lemma 2, Q is measurable, and

$$E^c \setminus Q \subset U_n \setminus E \;\; \forall n,$$

since if $x \in (E^c \setminus Q)$, $x \notin Q$, so $x \in Q^c = \bigcap_{i=1}^{\infty} U_n$, then $x \in U_n \forall n$, but $x \notin E$, so $x \in (U_n \setminus E)$. Then we have

$$m^*(E^c \setminus Q) \leq m^*(U_n \setminus E) < \frac{\varepsilon}{n} \leq \varepsilon$$

Since ε is arbitrary, $m^*(E^c \setminus Q) = 0$. Then by Lemma 1, $\forall \delta > 0$, $\exists$ open set $U \supset (E^c \setminus Q)$ s.t.

$$m^*(U) < m^*(E^c \setminus Q) + \delta = \delta$$

Since $(U \setminus (E^c \setminus Q)) \subset U$,

$$m^*(U \setminus (E^c \setminus Q)) < \delta \Rightarrow E^c \setminus Q \text{ is measurable.}$$

Lastly, since

$E^c = (E^c \setminus Q) \cup Q$, and both $E^c \setminus Q$ and Q are measurable,

we have E^c measurable.