Ex 3.

We will prove the general case $\mathbb{R}^k$, then the $k=2$ case follows.

Let $f(x_1, \dots, x_{n-1}) = x_n$ be the hyperplane, then, f is uniformly continuous on $[0,1]^{k-1}$. So, for $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{2}, \quad x,y \in [0,1]^{k-1}, \; d(x,y) = \|x-y\|_\infty$$

Then, fix n s.t. $\frac{1}{n} < \delta$, let

$$z = (z_1, \dots, z_{k-1}), \quad z_i \in \{0, 1, \dots, n-1\}$$

$$a_z = \frac{z}{n}, \quad B_z = \{(b_1, \dots, b_{k-1}) : \frac{z_i}{n} \leq b_i \leq \frac{z_i + 1}{n}\}$$

$$\bigcup_z B_z = [0,1]^{k-1}, \quad m(B_{z_i} \cap B_{z_j}) = 0 \text{ for } i \neq j$$

$$\Rightarrow 1 = m([0,1]^{k-1}) = m(\bigcup_z B_z) = \sum_z m(B_z)$$

We know that for $x \in B_z$,

$$d(a_z, x) < \frac{1}{n} < \delta \Rightarrow |f(a_z) - f(x)| < \frac{\varepsilon}{2}$$

Therefore,

$$E_z := \{(x, f(x)) : x \in B_z\} \subset B_z \times [f(a_z) - \frac{\varepsilon}{2}, f(a_z) + \frac{\varepsilon}{2}]$$

$$\Rightarrow m(E_z) \leq m(B_z \times [f(a_z) - \frac{\varepsilon}{2}, f(a_z + \frac{\varepsilon}{2})])$$

$$= m(B_z)\, m([f(a_z) - \frac{\varepsilon}{2}, f(a_z + \frac{\varepsilon}{2})])$$

$$= \varepsilon \cdot m(B_z)$$

Since $\bigcup_z E_z = \{(x, f(x)) : x \in [0,1]^{k-1}\}$,

$$m(\{(x, f(x)) : x \in [0,1]^{k-1}\}) \leq \sum_z \varepsilon \cdot m(B_z)$$

$$= \varepsilon \sum_z m(B_z)$$

$$= \varepsilon$$

$$\Rightarrow m(S := \{(x, f(x)) : x \in [0,1]^{k-1}\}) = 0$$

Since $\{x, f(x) : x \in \mathbb{R}^{k-1}\}$ can be covered by a countable union of S, $\{x, f(x) : x \in \mathbb{R}^{k-1}\}$ also has measure zero $\blacksquare$

Ex. 6

Thm. 16

$\Rightarrow$ Assume E is unbounded and measurable. Then, we know that E can be partitioned into a countable disjoint union of open boxes (plus a zero set that we will ignore).

$$E = \bigsqcup_{j=1}^{\infty} E_j \quad (\text{Lemma from lecture 6})$$

Since E_j is bounded and measurable, by the bounded version of this theorem, $\exists$ G_δ set G_j and F_σ set F_j s.t.

$$F_j \subset E_j \subset G_j \quad \text{and} \quad m(G_j \setminus F_j) = 0$$

We also now that

$$F := \bigcup_{j=1}^{\infty} F_j \text{ is also a } F_\sigma \text{ set}, \quad F \subset E \text{ and}$$

$$G := \bigcup_{j=1}^{\infty} G_j \text{ is also a } G_\delta \text{ set}, \quad E \subset G.$$

Lastly, since

$$G \setminus F \subset \bigcup_{j=1}^{\infty} (G_j \setminus F_j),$$

we have

$$m(G \setminus F) \le m\left(\bigcup_{j=1}^{\infty} (G_j \setminus F_j)\right) \le \sum_{j=1}^{\infty} m(G_j \setminus F_j) = 0$$

$\Leftarrow$ The proof for this direction in Pugh works for unbounded sets. $\blacksquare$

Thm 21

Let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^k$ be both measurable and unbounded.

Similar to Thm. 16,

$$A = \bigsqcup_{i=1}^{\infty} A_i, \quad B = \bigsqcup_{j=1}^{\infty} B_j$$

$$\Rightarrow A \times B = \bigsqcup_{i=1}^{\infty} A_i \times \bigsqcup_{j=1}^{\infty} B_j$$

$$= \bigsqcup_{i,j} (A_i \times B_j)$$

Therefore,

$$m(A \times B) = m(\bigsqcup_{i,j} m(A_i \times B_j))$$

$$= \sum_{i,j} m(A_i \times B_j)$$

$$= \sum_{i,j} m(A_i)\, m(B_j)$$

$$= \sum_{i} m(A_i) \sum_{j} m(B_j)$$

$$= m(\bigsqcup_{i} A_i)\, m(\bigsqcup_{j} B_j)$$

$$= m(A)\, m(B)$$

$\blacksquare$

Ex.12

First, since $A \subset \bar{A}$,

$J^*(A) \le J^*(\bar{A})$ by monotonicity

Then it suffices to show $J^*(\bar{A}) \le J^*(A)$. For $\varepsilon > 0$, $\exists \{B_j\}_{j=1}^N$ an open box covering of $J^*(A)$ s.t.

$$\sum_{j=1}^{N} |B_j| \le J^*(A) + \frac{\varepsilon}{2}$$

Now, since $\{B_j\}_{j=1}^N$ covers A, $\{\bar{B}_j\}_{j=1}^N$ covers $\bar{A}$ since

- For $x \in A$, $x \in B_j$ for some $j \Rightarrow x \in \bar{B}_j$
- For $x \in \bar{A} \setminus A$, x is a limit point of A. Then,

  $\mathcal{B}_\delta(x) \cap A \ne \phi$ $\forall \delta > 0$, where $\mathcal{B}_\delta(x)$ is an open ball with radius δ.

  $\Rightarrow \mathcal{B}_\delta(x) \cap B_j \ne \phi$ for some j $\forall \delta > 0$

  $\Rightarrow x$ is a limit point of B_j

  $\Rightarrow x \in \bar{B}_j$ by definition of closure

Therefore, we have

$$J^*(\bar{A}) \le \sum_{j=1}^{N} |\bar{B}_j| \qquad \left(\begin{array}{l}\text{Def. of } J^* \\ \text{11(d) that } B_j \text{ need not be open}\end{array}\right)$$

$$= \sum_{j=1}^{N} |B_j|$$

$$\le J^*(A) + \varepsilon$$

$\Rightarrow J^*(\bar{A}) \le J^*(A)$

$\Rightarrow J^*(\bar{A}) = J^*(A)$

Lastly, the equality $J^*(\bar{A}) = m(\bar{A})$ follows from 11(c) since $\bar{A}$ is closed and bounded, thus compact.