25. Let $f : \mathbb{R} \to [0, \infty)$ be given.

(a) If f is measurable why is the graph of f a zero set?

Let $G := \text{graph}(f) = \{(x, f(x)) : x \in \mathbb{R}\}$, then since f is a function, for every $x \in \mathbb{R}$, $\exists$ at most one $y \in [0,\infty)$ s.t. $y = f(x)$. Therefore, a slice G_x of G has at most one point. Therefore, $m(G_x) = 0$.

By the Zero Slice Theorem, then, $m(G) = 0$.

(b) If the graph of f is a zero set does it follow that f is measurable?

Let $E \subset \mathbb{R}$ be a nonmeasurable set, and let $f(x) = \mathbb{1}_E(x)$.

Then $\{(x, f(x)) : x \in \mathbb{R}\} \subset ((\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}))$, then since

$$m(\mathbb{R} \times \{0\}) + m(\mathbb{R} \times \{1\}) = 0 + 0 = 0,$$

$$\Rightarrow m(\text{graph}(f)) = 0$$

But E nonmeasurable $\Rightarrow$ f nonmeasurable, so the statement does not hold.

should be since $E \times [0,1]$ unmeasurable

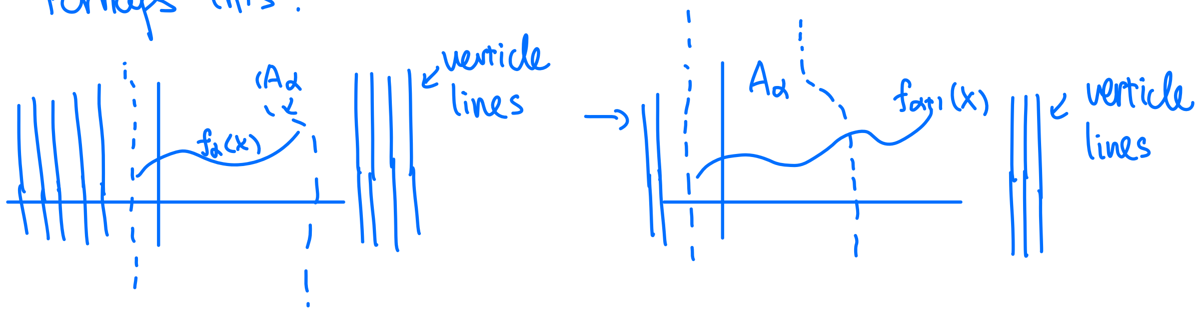
**(c) Read about transfinite induction and go to stackexchange to see that there exists a nonmeasurable function $f : [a, b] \to [0, \infty)$ whose graph is nonmeasurable.

Sketch of construction*:

- Enumerate G_δ sets in $\mathbb{R}^2$ as A_α for $\alpha < \aleph_1$
- Let $f : \mathbb{R} \to \mathbb{R}$. $\forall \alpha < \aleph_1$, $\exists f_\alpha$ an approximation to f s.t. f is defined only on α points.
- If we can add one more point to the domain of f_α s.t. $(x, f_{\alpha+1}(x))$ is outside A_α, then do so. If not, A_α contains the complement of countably many verticle lines in $\mathbb{R}^2 \Rightarrow A_\alpha$ has full measure. why?

* stackexchange subject: "Lebesgue Measure of the Graph of a Function" by Cosmonut on 4/28/2011

I want to draw a picture of this step, but not sure how. Perhaps this?



- Eventually, if we extend the function to f. The graph of f is not contained in any G_δ set with no full measure.

  $\Rightarrow$ graph(f) has full outer measure. How does this lead to f being a non-measurable function?

(d) Infer that the measurability hypothesis in the Zero Slice Theorem (Theorem 26) is necessary since every vertical slice graph of the function in (c) is a zero set (it is just a single point) and yet the graph has positive outer measure.

In Theorem 26, for a measurable set $E \subset \mathbb{R}^k \times \mathbb{R}^n$, if almost all slices of E is a zero set, E is a zero set. Thus

$$m(E) = 0 \Rightarrow m^*(E) = 0.$$

If we drop the measurability requirement, then, for the nonmeasurable function we constructed in (c), since each slice of its graph is a point in $\mathbb{R}^2$, it has measure zero. However, the graph of the function has positive outer measure.

(e) Why can a graph never have positive inner measure?

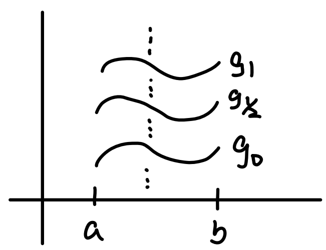
Let $G := \text{graph}(f)$. Then since f is a function, every slice G_x of G contains only one point, thus has measure zero

Now, consider any measurable subset $F \subset G$, its slices F_x will also only contain one point. So by the Zero Slice Theorem, $m(G) = 0$. Therefore G cannot have positive inner measure.

(f) How does (c) yield an example of uncountably many disjoint subsets of the plane, each with infinite outer measure?

The function f in (c) is $f : [a, b] \to [0, \infty)$, whose graph has a positive outer measure. Then, define $g_r(x) = f(x) + r$, $r \in \mathbb{R}$.

Then $[a, b] \times \mathbb{R} \subset \bigsqcup_{r \in \mathbb{R}} \text{graph}(g_r)$ (I think ?).



If we extend the domain of f to $\mathbb{R}$ to construct functions $\tilde{g}_r$, then

$$\text{graph}(g_r) \subset \text{graph}(\tilde{g}_r)$$

(Need to show: $m^*(\text{graph}(\tilde{g}_r)) = \infty$)

Then,

$$\mathbb{R}^2 \subset \bigsqcup_{r \in \mathbb{R}} \text{graph}(\tilde{g}_r)$$

(g) What assertion can you make from (f) and Exercise 19?

**19. Consider linear Lebesgue measure m_1 on the interval I and planar Lebesgue measure m_2 on the square I^2. Construct a meseometry $I \to I^2$. Thus meseometry disrespects topology: $(I, \mathcal{M}(I), m_1)$ is meseometric to $(I^2, \mathcal{M}(I^2), m_2)$. [Hint: You might use the following outline. The inclusion $I \setminus \mathbb{Q} \to I$ is injective and preserves m_1. You can convert it to a bijection $\alpha : I \setminus \mathbb{Q} \to I$ by choosing a countable set $L \subset I \setminus \mathbb{Q}$ and then choosing any bijection $\alpha_0 : L \to L \cup (\mathbb{Q} \cap I)$. Then you can set $\alpha(x) = \alpha_0(x)$ when $x \in L$ and $\alpha(x) = x$ otherwise. Why is α is a meseometry? (Already this shows that nonhomeomorphic spaces can have meseometric measure spaces.) In the same way there is a meseometry $\beta : I^2 \setminus \mathbb{Q}^2 \to I^2$. Then let $A = I \setminus \mathbb{Q}$. Express $x \in A$ as a base-2 expansion

$$x = (a_1 a_2 a_3 a_4 a_5 a_6 \dots)$$

using the digits 0 and 1. It is unique since x is irrational. Then consider the corresponding base-4 expansion

$$\sigma(x) = ((a_1 a_2)(a_3 a_4)(a_5 a_6) \dots)$$

using the digits (00), (01), (10), and (11). Prove that $\sigma(A) = I^2 \setminus \mathbb{Q}^2$ and σ preserves measure. Conclude that $T = \beta \circ \sigma \circ \alpha^{-1}$ is a meseometry $I \to I^2$.]

Idea: (19) shows that $\exists$ a meseometry $I \to I^2$.

From (f), we can send a subset of $\mathbb{R}$ to $\mathbb{R}^2$ (denoted E), while preserving measurability: take subsets of the translations of graph(f) (denoted $\bigcup_i F_i$) with the preimage equal to E. Then this algorithm that sends E to graph(f) is the meseometry.

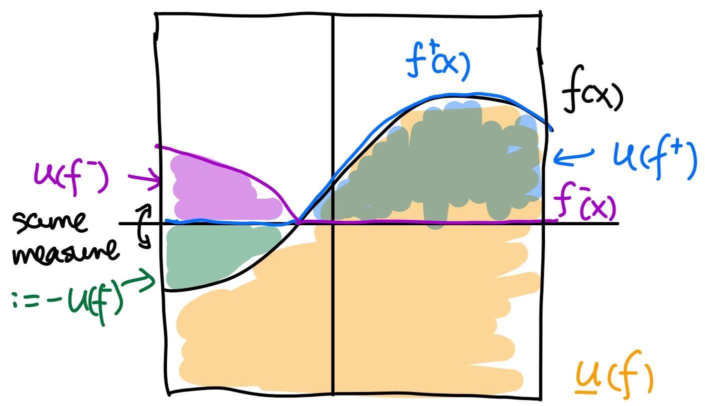
28. The **total undergraph** of $f:\mathbb{R}\to\mathbb{R}$ is $\underline{\mathcal{U}}f = \{(x,y) : y < f(x)\}$.

(a) Using undergraph pictures, show that the total undergraph is measurable if and only if the positive and negative parts of f are measurable.

The positive and negative parts of the function is

$f^+(x) = \max(f(x), 0)$

$f^-(x) = \max(-f(x), 0)$



Let $E^+ := \{(x,y) : x\in\mathbb{R}, y>0\}$

$E^- := \{(x,y) : x\in\mathbb{R}, y<0\}$

We know E^+, E^- measurable.

($\Rightarrow$) Assume $\underline{\mathcal{U}}(f)$ is measurable, then

$\mathcal{U}(f^+) = \underline{\mathcal{U}}(f) \cap E^+ \Rightarrow \mathcal{U}(f^+)$ measurable $\Rightarrow f^+$ measurable

$\mathcal{U}(f^-) = [\underline{\mathcal{U}}(f) \cup E^+]^c \Rightarrow \mathcal{U}(f^-)$ measurable $\Rightarrow f^-$ measurable

($\Leftarrow$) Assume f^+, f^- are measurable, then $\mathcal{U}(f^+), \mathcal{U}(f^-)$ measurable, then

$\underline{\mathcal{U}}(f) = \mathcal{U}(f^+) \cup ((-\mathcal{U}(f))^c \cap E^-) \Rightarrow \mathcal{U}(f)$ measurable

(b) Suppose that $f:\mathbb{R}\to(0,\infty)$ is measurable. Prove that $1/f$ is measurable. [Hint: The diffeomorphism $T:(x,y)\mapsto(x,1/y)$ sends $\underline{\mathcal{U}}f$ to $\underline{\mathcal{U}}(1/f)$.]

We know: $\mathcal{U}(f)$ is measurable, and want to show $\mathcal{U}(\frac{1}{f})$ is measurable.

From Ex. 23, we know that a diffeomorphism is a meseomorphism, thus preserves measurability.

Since $T: \mathcal{U}(f) \to \mathcal{U}(\frac{1}{f})$ is a diffeomorphism and $(x,y) \mapsto (x, \frac{1}{y})$

$\mathcal{U}(f)$ is measurable, $\mathcal{U}(\frac{1}{f})$ is measurable, so $\frac{1}{f}$ is measurable.

(c) Suppose that $f, g : \mathbb{R} \to (0, \infty)$ are measurable. Prove that $f \cdot g$ is measurable. [Hint: $T : (x, y) \mapsto (x, \log y)$ sends $\mathcal{U}f$ and $\mathcal{U}g$ to $\mathcal{U}(\log f)$ and $\mathcal{U}(\log g)$. How does this imply $\log fg$ is measurable, and how does use of $T^{-1} : (x, y) \mapsto (x, e^y)$ complete the proof?]

First it is clear that both T and T^{-1} are diffeomorphisms, so they both preserve measure. Then,

$$
\begin{aligned}
f, g \text{ measurable} &\Rightarrow u(f), u(g) \text{ measurable} \\
&\Rightarrow u(\log f), u(\log g) \text{ measurable} \\
&\Rightarrow \log f, \log g \text{ measurable} \\
&\Rightarrow \log f + \log g \text{ measurable}
\end{aligned}
$$

Next, we have, $\forall x$,

$$
\begin{aligned}
&\log f(x) + \log g(x) = \log[f(x) g(x)] \\
&\Rightarrow \log f + \log g = \log(fg) \\
&\Rightarrow u(\log f + \log g) = u(\log fg) \text{ measurable}
\end{aligned}
$$

Since T^{-1} preserves measure and sends $u(\log fg)$ to $u(fg)$, $u(fg)$ is measurable, thus fg is measurable.

(d) Remove the hypotheses in (a)-(c) that the domain of f, g is $\mathbb{R}$.

For (a), we can easily generalize to $\mathbb{R}^n$ since the set operations hold in higher dimensions.

For (b) and (c), we can find diffeomorphism T in higher dimensions.

(b): $T : (x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n, \frac{1}{y})$

(c): $T : (x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n, \log y)$

(e) Generalize (c) to the case that f, g have both signs.

For f, g that have both signs, we only need to divide $u(f)$ and $u(g)$ by the 4 quadrants. Then we can show that, individually, the 4 quadrants are measurable. Then the union of the pieces will be measurable.