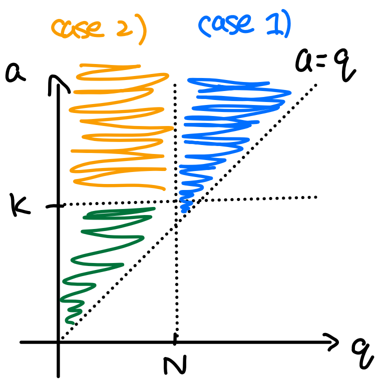
Exercise 8.2.7. Let $p > 2$ and $c > 0$. Using the Borel-Cantelli lemma, show that the set

$$\{x \in [0,1] : |x - \frac{a}{q}| \le \frac{c}{q^p} \text{ for infinitely many positive integers } a, q\}$$

has measure zero. (Hint: one only has to consider those integers a in the range $0 \le a \le q$ (why?). Use Corollary 11.6.5 to show that the sum $\sum_{q=1}^{\infty} \frac{c(q+1)}{q^p}$ is finite.)



Let $\Omega_{a,q} = [\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}] \cap [0,1]$.

To see why we only need to consider $0 \le a \le q$, consider the following cases:

1) $\exists N \in \mathbb{N}^+$ s.t. $\forall q > N$, $q^{p-1} > c$. Let $a > q > N$, then

$$\frac{a}{q} - \frac{c}{q^p} \geq \frac{q+1}{q} - \frac{c}{q^p} = 1 + \frac{1}{q} - \frac{c}{q^p} = 1 + \frac{q^{p-1} - c}{q^p} > 1$$

$$\Rightarrow \Omega_{a,q} = \phi$$

2) For $q < N$, if $a > q$, then $\exists k \in \mathbb{N}^+$ s.t. $\forall a > k$,

$$\frac{a}{q} > 1 + \frac{c}{q^p} \Rightarrow \frac{a}{q} - \frac{c}{q^p} > 1 \Rightarrow \Omega_{a,q} = \phi$$

Now, the green region consists of only finite number of a and q. So $\sum_{a,q \in \text{green}} m(\Omega_{a,q})$ is finite. We then only need to check that $\sum_{0 \le a \le q} m(\Omega_{a,q})$ is finite. We have

$$\sum_{q=1}^{\infty} \sum_{a=1}^{q} m(\Omega_{a,q}) \le \sum_{q=1}^{\infty} \sum_{a=1}^{q} m([\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}])$$

$$= \sum_{q=1}^{\infty} \sum_{a=1}^{q} \frac{2c}{q^p}$$

$$= \sum_{q=1}^{\infty} \frac{2c}{q^{p-1}} \qquad (1)$$

By Corollary 11.6.5, since $p > 2$, (1) converges.

Therefore, $\sum_{0\le a\le q} m(\Omega_{a,q}) < \infty$.

By the Borel–Canteli Lemma,

$$m(\{x\in[0,1]: |x-\frac{a}{q}| \le \frac{c}{q^p} \text{ for infinitely many } a,q\})$$

$$= m(\{x\in\mathbb{R}: x\in\Omega_{a,q} \text{ for infinitely many } a,q\})$$

$$= 0$$

Exercise 8.2.9. For every positive integer n, let $f_n : \mathbf{R} \to [0, \infty)$ be a non-negative measurable function such that

$$\int_{\mathbf{R}} f_n \leq \frac{1}{4^n}.$$

Show that for every $\varepsilon > 0$, there exists a set E of Lebesgue measure $m(E) \leq \varepsilon$ such that $f_n(x)$ converges pointwise to zero for all $x \in \mathbf{R} \backslash E$. (Hint: first prove that $m(\{x \in \mathbf{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}) \leq \frac{\varepsilon}{2^n}$ for all $n = 1, 2, 3, \ldots$, and then consider the union of all the sets $\{x \in \mathbf{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}$.)

Let $A_n = \{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}$, and define

$$h_n(x) = \frac{1}{\varepsilon 2^n} \mathbb{1}_{A_n}(x)$$

Since h_n is a simple function, and $h_n(x) \leq f_n(x) \ \forall x \in \mathbb{R}$,

$$\frac{1}{\varepsilon 2^n} m(A_n) = \int_{\mathbb{R}} \frac{1}{\varepsilon 2^n} \mathbb{1}_{A_n}(x) = \int_{\mathbb{R}} h_n(x) \leq \int_{\mathbb{R}} f_n(x) \leq \frac{1}{4^n}$$

$$\Rightarrow m(A_n) \leq \frac{\varepsilon}{2^n}$$

Let $E = \bigcup_{n=1}^{\infty} A_n$, then

$$m(E) \leq \sum_{n=1}^{\infty} m(A_n) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

If $x \in \mathbb{R} \backslash E$, then $f_n(x) \leq \frac{1}{\varepsilon 2^n} \ \forall n$. Since $\frac{1}{\varepsilon 2^n} \to 0$ as $n \to \infty$, $f_n(x) \to 0$ as well.

Exercise 8.2.10. For every positive integer n, let $f_n : [0,1] \to [0,\infty)$ be a non-negative measurable function such that f_n converges pointwise to zero. Show that for every $\varepsilon > 0$, there exists a set E of Lebesgue measure $m(E) \le \varepsilon$ such that $f_n(x)$ converges *uniformly* to zero for all $x \in [0,1]\backslash E$. (This is a special case of *Egoroff's theorem*. To prove it, first show that for any positive integer m, we can find an $N > 0$ such that $m(\{x \in [0,1] : f_n(x) > 1/m\}) \le \varepsilon/2^m$ for all $n \ge N$.) Is the claim still true if $[0,1]$ is replaced by $\mathbf{R}$?

Let $F_{n,m} = \{x \in [0,1] : f_n(x) > \frac{1}{m}\}$, and $G_{n,m} = \bigcup_{i=n}^{\infty} F_{i,m}$.

We know that $G_{n+1,m} \subset G_{n,m}$. Let $x \in G_{k,m}$ for some k,

then since $f_n(x) \to 0$ pointwise, $\exists M \in \mathbb{N}$ s.t.

$$\forall n > M, \quad f_n(x) < \frac{1}{m}$$

This means that for some $l > k$, $x \notin G_{l,m}$.

Therefore, $\bigcap_{n=1}^{\infty} G_{n,m} = \phi$. And

$$0 = m\left(\bigcap_{n=1}^{\infty} G_{n,m}\right) = \lim_{n\to\infty} m(G_{n,m}) \qquad (1)$$

Then, for each m, $\exists N^{(m)} \in \mathbb{N}$ s.t. $\forall n > N^{(m)}$,

$$m(G_{n,m}) \le \frac{\varepsilon}{2^m}$$

Let $n_m = \min\{n \in \mathbb{N} : n > N^{(m)}\}$, and $E = \bigcup_{m=1}^{\infty} G_{n_m,m}$, then

$$m(E) \le \sum_{m=1}^{\infty} m(G_{n_m,m}) \le \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = \varepsilon$$

If $x \in [0,1]\backslash E$, then $x \notin E \Rightarrow x \notin G_{n_m,m} \ \forall m$.

Therefore, $\forall p > n_m$, $f_p(x) \le \frac{1}{m} \Rightarrow f_n(x)$ converges to 0 uniformly.

If $[0,1]$ is switched to $\mathbb{R}$, the claim does not hold.

Loosely speaking, this is because (1) holds only when $m(G_{1,m}) < \infty$.

But once $G_{1,m}$ is not bounded, it can have infinite measure.

As a counterexample, consider $f_n(x) = \mathbb{1}_{[n, n+1]}(x)$ on $\mathbb{R}$.

$f_n(x) \to 0$ pointwise, but it does not converge uniformly.