Exercise 8.3.2. Prove Proposition 8.3.3. (Hint: for (b), break f, g, and $f+g$ up into positive and negative parts, and try to write everything in terms of integrals of non-negative functions only, using Lemma 8.2.10.)

Proposition 8.3.3. *Let Ω be a measurable set, and let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be absolutely integrable functions.*

(a) For any real number c (positive, zero, or negative), we have that cf is absolutely integrable and $\int_\Omega cf = c\int_\Omega f$.

(b) The function $f+g$ is absolutely integrable, and $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$.

(c) If $f(x) \le g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \le \int_\Omega g$.

(d) If $f(x) = g(x)$ for almost every $x \in \Omega$, then $\int_\Omega f = \int_\Omega g$.

Proof. See Exercise 8.3.2. □

(a) Since f is absolutely integrable,

$$\int_\Omega |f| < \infty$$

$$\Rightarrow \int_\Omega |cf| = \int_\Omega |c||f|$$

$$= |c| \int_\Omega |f| \quad \text{(By Proposition 8.2.6)}$$

$$< \infty$$

$\Rightarrow cf$ is absolutely integrable

Now, since $\int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^-$ by definition

we have 3 cases:

1) $c=0$: $\int_\Omega cf = \int_\Omega 0 = 0 = c\int_\Omega f$

2) $c>0$: $\int_\Omega (cf)^+ = \int_\Omega cf^+ = c\int_\Omega f^+$

$\int_\Omega (cf)^- = \int_\Omega cf^- = c\int_\Omega f^-$

$$\Rightarrow \int_\Omega cf = c\int_\Omega f^+ - c\int_\Omega f^-$$

$$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right)$$

$$= c\int_\Omega f$$

3) $c<0$: $\int_\Omega (cf)^+ = \int_\Omega \max(cf, 0)$

$$= \int_\Omega c\min(f,0)$$

$$= \int_\Omega -cf^-$$

$$= -c\int_\Omega f^-$$

$\int_\Omega (cf)^- = \int_\Omega -\min(cf, 0)$

$$= \int_\Omega -c\max(f,0)$$

$$= \int_\Omega -cf^+$$

$$= -c\int_\Omega f^+$$

$$\Rightarrow \int_\Omega cf = -c\int_\Omega f^- + c\int_\Omega f^+$$

$$= c\left(\int_\Omega f^+ - \int_\Omega f^-\right)$$

$$= c\int_\Omega f$$

(b) First, since $\int_\Omega |f| < \infty$, $\int_\Omega |g| < \infty$, and $|f+g| \le |f| + |g|$,

$$\underset{\substack{\uparrow \\ \text{(Prop. 8.2.6(c))}}}{\int_\Omega |f+g| \le} \int_\Omega |f|+|g| \underset{\substack{\uparrow \\ \text{(Lemma 8.2.10)}}}{=} \int_\Omega |f| + \int_\Omega |g| < \infty$$

Then, we have

$$\left.\begin{array}{l} f = f^+ - f^- \\ g = g^+ - g^- \end{array}\right\} \Rightarrow f+g = f^+ - f^- + g^+ - g^-$$

$$= (f^+ + g^+) - (f^- + g^-)$$

Since, also, $f+g = (f+g)^+ - (f+g)^-$,

$$(f+g)^+ - (f+g)^- = (f^+ + g^+) - (f^- + g^-)$$

$$\Rightarrow (f+g)^+ + (f^- + g^-) = (f^+ + g^+) + (f+g)^-$$

$$\Rightarrow \int_\Omega \left[(f+g)^+ + (f^- + g^-)\right] = \int_\Omega \left[(f^+ + g^+) + (f+g)^-\right] \quad \text{(Prop. 8.2.6(d))}$$

$$\Rightarrow \int_\Omega (f+g)^+ + \int_\Omega (f^- + g^-) = \int_\Omega (f^+ + g^+) + \int_\Omega (f+g)^-$$

$$\Rightarrow \int_\Omega (f+g)^+ - \int_\Omega (f+g)^- = \int_\Omega (f^+ + g^+) - \int_\Omega (f^- + g^-)$$

Since, by definition, $\int_\Omega (f+g) = \int_\Omega (f+g)^+ - \int_\Omega (f+g)^-$,

$$\begin{aligned}
\int_\Omega (f+g) &= \int_\Omega (f^+ + g^+) - \int_\Omega (f^- + g^-) \\
&= \int_\Omega f^+ + \int_\Omega g^+ - \int_\Omega f^- - \int_\Omega g^- \\
&= \left(\int_\Omega f^+ - \int_\Omega f^-\right) + \left(\int_\Omega g^+ - \int_\Omega g^-\right) \\
&= \int_\Omega f + \int_\Omega g
\end{aligned}$$

(c) Since $f \le g$,

$$f^+ = \max(f, 0) \le \max(g, 0) = g^+$$

$$f^- = -\min(f, 0) \ge -\min(g, 0) = g^-$$

as we can consider the 3 cases : $\begin{cases} 0 \le f \le g \\ f \le 0 \le g \\ f \le g \le 0 \end{cases}$

Then,

$$\int_\Omega f^+ \le \int_\Omega g^+, \quad \int_\Omega f^- \ge \int_\Omega g^- \quad \text{(Prop. 8.2.6 (c))}$$

$$\Rightarrow \int_\Omega f = \int_\Omega f^+ - \int_\Omega f^- \le \int_\Omega g^+ - \int_\Omega g^- = \int_\Omega g$$

(d) Since $f(x) = g(x)$ for a.e. $x \in \Omega$,

$$f^+(x) = g^+(x), \quad f^-(x) = g^-(x) \text{ for a.e. } x \in \Omega$$

$$\Rightarrow \int_\Omega f = \int_\Omega f^+ - \int_\Omega f^- \underset{\text{(Prop. 8.2.6 (d))}}{=} \int_\Omega g^+ - \int_\Omega g^- = \int_\Omega g$$

Exercise 8.3.3. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be absolutely integrable, measurable functions such that $f(x) \leq g(x)$ for all $x \in \mathbf{R}$, and that $\int_{\mathbf{R}} f = \int_{R} g$. Show that $f(x) = g(x)$ for almost every $x \in \mathbf{R}$ (i.e., that $f(x) = g(x)$ for all $x \in \mathbf{R}$ except possibly for a set of measure zero).

By prop. 8.3.3 (a), (b)

$$\int_{\mathbb{R}} (g-f) = \int_{\mathbb{R}} g - \int_{\mathbb{R}} f = 0$$

Since $g \geq f$, $g-f \geq 0 \; \forall x \in \mathbb{R} \Rightarrow \int_{\mathbb{R}} (g-f) \geq 0$, by Prop. 8.2.6 (a),

$$\int_{\mathbb{R}} (g-f) = 0 \Rightarrow (g-f)(x) = g(x) - f(x) = 0 \text{ for a.e. } x \in \mathbb{R}$$

$$\Rightarrow g(x) = f(x) \text{ for a.e. } x \in \mathbb{R}$$