1. 39. Suppose that f and g are measurable and their squares are integrable. Prove that fg is measurable, integrable, and

$$\int fg \le \sqrt{\int f^2}\sqrt{\int g^2}.$$

[Hint: Exercise 28 helps.]

28. The **total undergraph** of $f : \mathbb{R} \to \mathbb{R}$ is $\underline{\mathcal{U}}f = \{(x, y) : y < f(x)\}$.
   (a) Using undergraph pictures, show that the total undergraph is measurable if and only if the positive and negative parts of f are measurable.
   (b) Suppose that $f : \mathbb{R} \to (0, \infty)$ is measurable. Prove that $1/f$ is measurable. [Hint: The diffeomorphism $T : (x, y) \mapsto (x, 1/y)$ sends $\mathcal{U}f$ to $(\widehat{\mathcal{U}}(1/f))^c$.]
   (c) Suppose that $f, g : \mathbb{R} \to (0, \infty)$ are measurable. Prove that $f \cdot g$ is measurable. [Hint: $T : (x, y) \mapsto (x, \log y)$ sends $\mathcal{U}f$ and $\mathcal{U}g$ to $\underline{\mathcal{U}}(\log f)$ and $\underline{\mathcal{U}}(\log g)$. How does this imply $\log fg$ is measurable, and how does use of $T^{-1} : (x, y) \mapsto (x, e^y)$ complete the proof?]
   (d) Remove the hypotheses in (a)-(c) that the domain of f, g is $\mathbb{R}$.
   (e) Generalize (c) to the case that f, g have both signs.

By Exercise 28 (c) and (e), $f \cdot g$ is measurable.

Idea: Use the quadratic trick for proving Cauchy-Schwarz

Let $A = \int fg$, $B = \int f^2$, $C = \int g^2$, $t > 0$

$\int (tf + g)^2 \ge 0$

$\Rightarrow \int (tf)^2 + g^2 + (2tfg) \ge 0$

$\Rightarrow t^2 \int f^2 + \int g^2 + 2t \int fg \ge 0$

$\Rightarrow Bt^2 + 2At + C \ge 0$

$\Rightarrow (2A)^2 - 4BC \le 0$

$\Rightarrow 4A^2 \le 4BC$

$\Rightarrow A^2 \le BC$

$\Rightarrow A \le \sqrt{B}\sqrt{C}$

2. *48. The Devil's ski slope. Recall from Chapter 3 that the Devil's staircase function $H : [0,1] \to [0,1]$ is continuous, nondecreasing, constant on each interval complementary to the standard Cantor set, and yet is surjective. For $n \in \mathbb{Z}$ and $x \in [0,1]$ we define $\widehat{H}(x+n) = H(x) + n$. This extends H to a continuous surjection $\mathbb{R} \to \mathbb{R}$. Then we set

$$H_k(x) = \widehat{H}(3^k x) \quad \text{and} \quad J(x) = \sum_{k=0}^{\infty} \frac{H_k(x)}{4^k}.$$

Prove that J is continuous, strictly increasing, and yet $J' = 0$ almost everywhere. [Hint: Fix $a > 0$ and let

$$S_a = \{x : J'(x) \text{ exists, } J'(x) > a, \text{ and } x \text{ belongs to the constancy intervals of every } H_k\}.$$

Use the Vitali Covering Lemma to prove that $m^*(S_a) = 0$.]

- Show J is continuous.

First we note that since H is nondecreasing, $\hat{H}$ is nondecreasing.
For $x \in [0,1]$, $3^k x \le 3^k$ $\forall k \in \mathbb{Z}^*$, therefore,

$$\hat{H}(3^k x) \le \hat{H}(3^k) = 3^k \quad \forall k \in \mathbb{Z}^*$$

Now, by Weierstrass M-test,

$$\sum_{k=0}^{\infty} \frac{3^k}{4^k} \text{ converges} \Rightarrow \sum_{k=0}^{\infty} \frac{\hat{H}(3^k x)}{4^k} \text{ converges uniformly}$$

Since H is continuous, $\hat{H}$ is continuous, and then H_k is continuous, thus $\frac{H_k(x)}{4^k}$ is continuous $\forall k$ $\forall x \in [0,1]$. Therefore, J is continuous on $[0,1]$.

- Show J is strictly increasing.

We know that at stage k of constructing the Cantor Function, if for $x, y \in [0,1]$, $y - x > \frac{1}{3^n}$, y and x would lie in different intervals of constant values, and $H(y) > H(x)$. Thus $\hat{H}(y) > \hat{H}(x)$, and $\hat{H}(3^k y) > \hat{H}(3^k x)$. And $\forall m > k$, $\hat{H}(3^m y) > \hat{H}(3^m x)$.
Since $\frac{1}{3^n} \to 0$, for $y > x$, we can always find k s.t. $y - x > \frac{1}{3^k}$

Therefore, $J(y) > J(x)\ \forall\ 0 \le x < y \le 1$. Thus J is strictly increasing.

- Show $J' = 0$ a.e.

Here's a sketch of my attemp:

1) Form a Vitali Cover of S_a by Cubes $\{Q_i\}$. Choose Q_k to be small s.t. each Q_k is contained in a constancy interval that contains a point in S_a for all k. This means that for $x \in Q_i$, $H_k(x)$ is constant $\forall k$.
2) Use Vitali Covering Lemma to reduce $\{Q_i\}$ to a countable subcollection $Q_1, Q_2, \dots$
3) By how we constructed S_a, each point x in $(S_a \cap Q_i)$ satisfies

$$\lim_{h \to 0} \frac{J(x+h) - J(x)}{h} > a$$

Since J strictly increases, $J(x+h) - J(x)$ decreases as h decreases. This means that $\frac{J(x+h) - J(x)}{h} > a\ \forall\ h > 0$.

And this is where I'm stuck as I am not sure how to turn this consideration into measures of the cubes.

3.

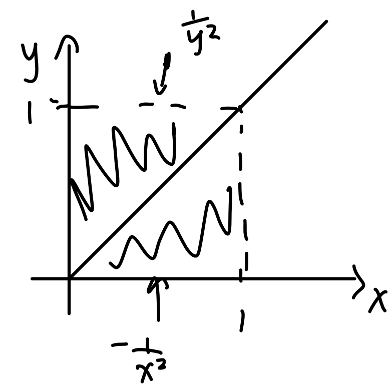
53. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{1}{y^2} & \text{if } 0 < x < y < 1 \\ \dfrac{-1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the iterated integrals exist and are finite (calculate them) but the double integral does not exist.

(b) Explain why (a) does not contradict Corollary 43.

(a)



$$\int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy$$

$$= \int_0^1 \left[\int_0^y f(x,y)\,dx + \int_y^1 f(x,y)\,dx\right] dy$$

$$= \int_0^1 \left[\int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx\right] dy$$

$$= \int_0^1 \left[\frac{x}{y^2}\Big|_0^y + \frac{1}{x}\Big|_y^1\right] dy$$

$$= \int_0^1 \left(\frac{1}{y} + 1 - \frac{1}{y}\right) dy$$

$$= \int_0^1 1\,dy = 1$$

$$\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx = \int_0^1 \left[\int_0^x -\frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dy\right] dx$$

$$= \int_0^1 \left[-\frac{y}{x^2}\Big|_0^x - \frac{1}{y}\Big|_x^1\right] dx$$

$$= \int_0^1 \left(-\frac{1}{x} - 1 + \frac{1}{x}\right) dx$$

$$= \int_0^1 -1\,dx = -1$$

At this point, if the double integral exists, it must equal to both interated integrals. But the interated integrals take on different values, this implies that the double integral does not exist.

(b) Corollary 43 requires f to be nonnegative, but here f can output negative values.

58. The balanced density of a measurable set E at x is the limit, if exists, of the concentration of E in B where B is a ball centered at x that shrinks down to x. Write $\delta_{\text{balanced}}(x, E)$ to indicate the balanced density, and if it is 1, refer to x as a balanced density point.

(a) Why is it immediate from the Lebesgue Density Theorem that almost every point of E is a balanced density point?

(b) Given $\alpha \in [0, 1]$, construct an example of a measurable set $E \subset \mathbb{R}$ that contains a point x with $\delta_{\text{balanced}}(x, E) = \alpha$.

(c) Given $\alpha \in [0, 1]$, construct an example of a measurable set $E \subset \mathbb{R}$ that contains a point x with $\delta(x, E) = \alpha$.

**(d) Is there a single set that contains points of both types of density for all $\alpha \in [0, 1]$?

$$\delta_b(x, E) = \lim_{r \to 0} \frac{m(B_r(x) \cap E)}{m(E)}$$

(a) $\forall\, x \in dp(E)$, for every cube that contains x, we can find a ball that is contained in that cube. This make every $x \in dp(E)$ also a balanced density point.

(b) I'm not sure how to construct this set directly, but here are my considerations:

- I want to prove that $\delta_{\text{balanced}}(0, E) = \alpha$ since it should be easier to construct symmetric intervals around zero, which could interact nicely as the ball shrinks.
- I want E to be a subset of $[-1, 1]$. Since $0 \leq \alpha \leq 1$, if we have to compute products of α and measures of subsets of E, we might want the product to stay between 0 and 1.
- E should contain intervals that shrinks as they get closer to 0. This is to ensure that as our $B_r(0)$ shrinks, $B_r(0)$ shrinks as well. The interval sizes should also be related to α.

After some research, I discovered a dissertation (Martin, Nathaniel F.G., "Metric density of sets" 1959.) that constructs such an interval. I describe the construction here:

For $n \in \mathbb{N}$, let

$$I_n^R = \left(\frac{1}{n+1}, \frac{1}{n}\right), \quad I_n^L = \left(-\frac{1}{n}, -\frac{1}{n+1}\right)$$

Let G_n^R, G_n^L be open sets s.t.

$$G_n^R \subset I_n^R, \quad m(G_n^R) = \alpha m(I_n^R)$$

$$G_n^L \subset I_n^L, \quad m(G_n^L) = \alpha m(I_n^L)$$

Finally, define

$$G = \bigcup_{n=1}^{\infty} (G_n^R \cup G_n^L)$$

The paper proved that $\delta_{\text{balanced}}(0, G) = \alpha$

(c) The same construction shown in (b) actually proves $\delta(0, G) = \alpha$ since the paper uses density, not balanced density.

(d) My initial idea is to extend the set constructed in (b) to $\mathbb{R}$, with a set contained in a interval with length 2, whose density at the midpoint of the interval is α. But this set is unbounded and should have infinite measure.

I leave a reference here that constructs the required interval.

I will update my homepage once I understand the proof.

If we end up doing final papers, I feel like topics related to this could be interesting.

deCamp, Allan, "The Construction of a Lebesgue Measurable Set with Every Density", Real Analysis, Vol 16(1), 1990-91, pg 344-348.

66. Construct a monotone function $f : [0,1] \to \mathbb{R}$ whose discontinuity set is exactly the set $\mathbb{Q} \cap [0,1]$, or prove that such a function does not exist.

This proof is adapted from "a monotonic function whose points of discontinuity form a dense set" on mathcounterexamples.net, where a general case (with a generic dense subset) is shown.

Let $D = \mathbb{Q} \cap [0,1]$, D is dense in $[0,1]$. We label the elements of D as $d_1, d_2, \dots$. Let $f(x) = \sum_{\{n : d_n \le x\}} (\frac{1}{2})^n$.

- Show f is increasing.

  Let $0 \le x < y \le 1$, then

  $$f(y) - f(x) = \sum_{\{n : x < d_n \le y\}} (\tfrac{1}{2})^n > 0 \text{ since } D \text{ is dense.}$$

- Show f is discontinuous $\forall x \in D$

  Let $x = d_m \in D$, $0 \le y < x$, then,

  $$f(x) - f(y) = \sum_{\{n : y < d_n \le x\}} (\tfrac{1}{2})^n \ge (\tfrac{1}{2})^m$$

  Therefore, f is not left-continuous $\forall x \in D$, thus f is discontinuous $\forall x \in D$.

- Show f is continuous $\forall x \in [0,1] \setminus D$

  Let $x \in [0,1] \setminus D$, $\forall \varepsilon > 0$, there exists...

  1) $N \in \mathbb{N}$ s.t. $0 < \sum_{n > N} (\frac{1}{2})^n < \varepsilon$. Let $\delta_1 > 0$ be small s.t. $(x, x+\delta_1) \cap \{d_1, d_2, \dots, d_N\} = \emptyset$

     Then, for $y - x < \delta_1$,

     $$0 < f(y) - f(x) \le \sum_{\{n : x < d_n \le y\}} (\tfrac{1}{2})^n \le \sum_{n > N} (\tfrac{1}{2})^n < \varepsilon$$

2) $D_0 \subset \{d_n \in D : d_n < x\}$ s.t. $\sum_{\{n : d_n \in D_0\}} (\frac{1}{2})^n > f(x) - \varepsilon$.

Let $\delta_2 > 0$ be small s.t. $(x - \delta_2, x) \cap D_0 = \emptyset$.

Then, for $x - y < \delta_2$

$$f(x) - \varepsilon < \sum_{\{n : d_n \in D_0\}} (\tfrac{1}{2})^n < \sum_{\{n : d_n \leq y\}} (\tfrac{1}{2})^n = f(y)$$

since $\{n : d_n \in D_0\} \subset \{n : d_n \leq y\}$.

$\Rightarrow f(x) - f(y) < \varepsilon$

Choose $\delta = \min(\delta_1, \delta_2)$, then

$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Thus f is continuous on $[0,1] \setminus D$