83. Suppose that (f_k) is a sequence of measurable functions that converge almost everywhere to f as $k \to \infty$.

(a) Formulate and prove Egoroff's Theorem if the functions are defined on a box in n-space.

(b) Is Egoroff's Theorem true or false for a sequence of functions defined on an an unbounded set having finite measure?

(c) Give an example of a sequence of functions defined on $\mathbb{R}$ for which Egoroff's Theorem fails.

(d) Prove that if the functions are defined on $\mathbb{R}^n$ and $\epsilon > 0$ is given then there is an ϵ-set $S \subset \mathbb{R}^n$ such that for each compact $K \subset \mathbb{R}^n$, the sequence of functions restricted to $K \cap S^c$ converges uniformly.

(a) Egoroff: Let $B^n = \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n$, then for $f_k : B \to \mathbb{R}$ a sequence of functions s.t. $f_k \to f$ as $k \to \infty$, if f_k converges almost everywhere, $f_k \to f$ uniformly nearly everywhere.

proof:

Essentially, we simply change the interval $[a, b]$ to the box B^n in the proof from the appendix.

Let

$$X(k, \ell) = \{x \in B^n : \forall n \geq k, \ |f_n(x) - f(x)| < \tfrac{1}{\ell}\}$$

For $\ell \in \mathbb{N}$, $f_n(x) \to f(x)$ a.e. implies that

$$\bigcup_k X(k, \ell) \cup Z(\ell) = B^n$$

Let $\varepsilon > 0$, we have

$$m(X(k, \ell)) \to m(B^n) = \prod_{i=1}^{n} (b_i - a_i) \quad \text{as } k \to \infty$$

Thus $\exists\ k_1 < k_2 < \cdots$ s.t.

$$m([X(k_\ell, \ell)]^c) < \frac{\varepsilon}{2^\ell} \Rightarrow m(X^c) < \varepsilon, \quad X^c = \bigcap_\ell X(k_\ell, \ell)$$

Then $\forall\ \sigma > 0$, fix $\frac{1}{\ell} < \sigma$, then $\forall\ n \geq k_\ell$,

$$x \in X \Rightarrow x \in X(k_\ell, \ell) \Rightarrow |f_k(x) - f(x)| < \tfrac{1}{\ell} < \sigma$$

Therefore, $\forall\ x \in X$, $f_k(x) \to f(x)$ uniformly.

(b) Yes. For an unbounded set with finite measure, we can keep a bounded set s.t. the rest of the set has small measure. We can make the bounded set's measure large enough s.t. the part we throw away, combined with X^c, has measure $< \varepsilon$. Then the proof of (a) works.

(c) Let $f_n(x) = \mathbb{1}_{[n,n+1]}$. $f_n(x) \to 0$ pointwise.

But $\forall n \in \mathbb{N}$, $x \in \mathbb{R}$

$$|f_n(x) - 0| = 1 > \varepsilon\text{-small}$$

Therefore, f_n does not converge to f uniformly.

(d) For each K compact, $\exists \{B_i\}$ a finite box covering of K s.t.

$$K \subset \bigcup_{i=1}^{m} B_i$$

On each B_i, use (a). Thus for each B_i, $\exists E_i$ s.t.

$$m(E_i) \text{ small (say} < \tfrac{\varepsilon}{2^i}) \text{ and } f_n \to f \text{ uniformly on } (K \cap (E_i^c \cap B_i))$$

$\Rightarrow \exists n_i \in \mathbb{N}$ s.t. $\forall n > n_i$, fix $\sigma > 0$

$$x \in K \cap E_i^c \Rightarrow |f_n(x) - f(x)| < \sigma$$

Let $N = \max\{n_1, \dots, n_m\}$, then

$$x \in \bigcup_{i=1}^{m} (K \cap E_i^c) \Rightarrow |f_n(x) - f(x)|$$

Therefore, let $S = \bigcup_{i=1}^{m} E_i$. then for $x \in K \cap S^c$,

$$n > N \Rightarrow |f_n(x) - f(x)| < \sigma, \text{ and } m(S) < \varepsilon$$

3. Let $(\mathbb{R}^n, |\cdot|_1)$ be the normed vector space where $|(x_1, \cdots, x_n)|_1 := \sum_i |x_i|$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, given by the matrix T_{ij}, that sends (x_i) to (y_j), where $y_i = \sum_j T_{ij} x_j$. How to compute $\|T\|$?

- optional: if we use $\| - \|_{max}$ norm on $\mathbb{R}^n$, how to compute the operator norm $\|T\|$?

Claim: $\sup\left\{\frac{|Tx|_1}{|x|_1} : x \neq 0\right\} = \sup\{|Tx|_1 : |x|_1 = 1\}$

proof: If $x \neq 0$, $\frac{1}{|x|_1}$ is well-defined, so

$$\frac{|Tx|_1}{|x|_1} = |Tx|_1\left(\frac{1}{|x|_1}\right) = \left|Tx\left(\frac{1}{|x|_1}\right)\right|_1 = \left|T\left(\frac{x}{|x|_1}\right)\right|_1$$

Then we notice $\left|\frac{x}{|x|_1}\right|_1 = 1$, so

$$\left\{\frac{|Tx|_1}{|x|_1} : x \neq 0\right\} = \{|Tx|_1 : |x|_1 = 1\}$$

So our problem becomes computing $\sup\{|Tx|_1 : |x|_1 = 1\}$

Let $T_1, \ldots, T_n$ denote the nth column of T. Then,

$$Tx = \begin{bmatrix} \sum_{j=1}^n T_{1j} x_j \\ \vdots \\ \sum_{j=1}^n T_{nj} x_j \end{bmatrix} = \begin{bmatrix} T_{11} x_1 \\ \vdots \\ T_{n1} x_1 \end{bmatrix} + \cdots + \begin{bmatrix} T_{1n} x_n \\ \vdots \\ T_{nn} x_n \end{bmatrix}$$

$$= \sum_{i=1}^n x_i T_i$$

Therefore,

$$\begin{aligned}
|Tx|_1 &= \left|\sum_{i=1}^n x_i T_i\right|_1 \\
&\leq \sum_{i=1}^n |x_i T_i|_1 \\
&= \sum_{i=1}^n |x_i| \, |T_i|_1 \qquad \text{(notice the difference of norm and absolute value)} \\
&\leq \left(\max_{1 \leq i \leq n} |T_i|_1\right) |x|_1 \\
&= \max_{1 \leq i \leq n} |T_i|_1
\end{aligned}$$

Let $k = \operatorname{argmax}_{1 \leq i \leq n} |T_i|_1$, then $|Tx|_1 = \max_{1 \leq i \leq n} |T_i|_1$ when $x =$ k-th standard basis of $\mathbb{R}^n$.

Thus $|T|_1 = \sup |Tx|_1 = \max_{1 \le i \le n} |T_i|_1$

If we use the max norm, the claim still holds, and for $|x|_{max} = 1$, then

$$|Tx|_{max} = \max_{1 \le i \le n} |y_i|$$
$$= \max_{1 \le i \le n} \left| \sum_{j=1}^{n} T_{ij} x_j \right|$$
$$\le \max_{1 \le i \le n} \sum_{j=1}^{n} |T_{ij} x_j|$$
$$= \max_{1 \le i \le n} \sum_{j=1}^{n} |T_{ij}| |x_j|$$
$$= \max_{1 \le i \le n} |x|_{max} \sum_{j=1}^{n} |T_{ij}|$$
$$= \max_{1 \le i \le n} \sum_{j=1}^{n} |T_{ij}|$$
$$= \max_{1 \le i \le n} |\tilde{T}_i|_1 \text{, where } \tilde{T}_i \text{ is the ith row of } T$$

Choose x s.t. $T_{ij} x_j = |T_{ij}|$, then

$$|T|_{max} = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} T_{ij} x_j \right|$$
$$= \max_{1 \le i \le n} \left| \sum_{j=1}^{n} |T_{ij}| \right|$$
$$= \max_{1 \le i \le n} |\tilde{T}_j|_1$$

So equality can be achieved, therefore,

$$|T|_{max} = \max_{1 \le i \le n} |\tilde{T}_i|_1 \text{, where } \tilde{T}_i \text{ is the ith row of } T$$

4. <u>Definition</u>: (Concave Functions) $f:[a,b]\to\mathbb{R}$ is said to be concave if

$\forall x, y \in [a,b]$, $t \in [0,1]$,

$$f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$$

<u>Lemma 1</u>: (Young's Inequality)

If $a \geq 0$, $b \geq 0$, $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

<u>proof</u>: (From Wikipedia)

Let $t = \frac{1}{p}$ then $1-t = \frac{1}{q}$. Since the log function is concave, and $\frac{1}{p}, \frac{1}{q} \in [0,1]$, we have

$$\begin{aligned} \log(ta^p + (1-t)b^q) &\geq t\log(a^p) + (1-t)\log(b^p) \\ &= \log(a) + \log(b) \\ &= \log(ab) \end{aligned}$$

$$\Rightarrow \frac{a^p}{p} + \frac{b^q}{q} = ta^p + (1-t)b^q \geq ab$$

<u>Theorem 1</u>: (Hölder's Inequality)

If $p \geq 1, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

<u>proof</u>: (From Math StackExchange post answered by Soap)

The case is trivial if $\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} = 0$ and/or $\left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} = 0$.

Assume the 2 quantities above are both nonzero, then let

$$z_k = \frac{x_k}{\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}}, \quad w_k = \frac{y_k}{\left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}}$$

Then, by Young's Inequality,

$$\sum_{k=1}^{n} |z_k w_k| = \sum_{k=1}^{n} |z_k||w_k| \leq \sum_{k=1}^{n} \left(\frac{|z_k|^p}{p} + \frac{|w_k|^q}{q} \right)$$

We also have

$$\sum_{l=1}^{n} |z_l|^p = \sum_{l=1}^{n} \frac{|x_l|^p}{\left|\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}\right|^p} = \frac{\sum_{l=1}^{n} |x_l|^p}{\left(\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}\right)^p} = 1$$

Similarly,

$$\sum_{l=1}^{n} |w_l|^q = 1$$

Therefore,

$$\sum_{k=1}^{n} \left(\frac{|z_k|^p}{p} + \frac{|w_k|^q}{q} \right) = \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \sum_{k=1}^{n} |z_k w_k| \leq 1$$

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \sum_{k=1}^{n} |z_k w_k| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

$$\Rightarrow \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \sum_{k=1}^{n} \left(\frac{|x_k|}{\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}} \cdot \frac{|y_k|}{\left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}} \right)$$

$$= \sum_{k=1}^{n} |x_k||y_k| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

∎

Theorem 2: (Minkowski's Inequality)

For any $p \geq 1$,

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

proof: (From comm.utoronto.ca/frank/notes/ineq.pdf)

First,

$$\sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n}\left(|x_i + y_i|\,|x_i + y_i|^{p-1}\right)$$

$$\leq \sum_{i=1}^{n}\left[(|x_i| + |y_i|)\,|x_i + y_i|^{p-1}\right] \quad \text{(Triangle Inequality)}$$

$$= \underbrace{\sum_i |x_i|\,|x_i + y_i|^{p-1}}_{(1)} + \underbrace{\sum_i |y_i|\,|x_i + y_i|^{p-1}}_{(2)}$$

Let $q = \frac{p}{p-1}$, thus $\frac{1}{p} + \frac{1}{q} = 1$, and by Hölder's Inequality,

$$(1) \leq \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} \left(\sum_i |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}}$$

$$= \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} \left(\sum_i |x_i + y_i|^p\right)^{\frac{1}{q}}$$

$$(2) \leq \left(\sum_i |y_i|^p\right)^{\frac{1}{p}} \left(\sum_i |x_i + y_i|^p\right)^{\frac{1}{q}}$$

Therefore,

$$\sum_{i=1}^{n} |x_i + y_i|^p \leq \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} \left(\sum_i |x_i + y_i|^p\right)^{\frac{1}{q}} + \left(\sum_i |y_i|^p\right)^{\frac{1}{p}} \left(\sum_i |x_i + y_i|^p\right)^{\frac{1}{q}}$$

$$= \left[\left(\sum_i |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |y_i|^p\right)^{\frac{1}{p}}\right] \left(\sum_i |x_i + y_i|^p\right)^{\frac{p-1}{p}}$$

$$\Rightarrow \left(\sum_i |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_i |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_i |y_i|^p\right)^{\frac{1}{p}} \quad \left(\text{since } \left(\sum_i |x_i + y_i|^p\right)^{\frac{p-1}{p}} > 0\right)$$

$\boxtimes$