1. **6.** If $f(0,0)=0$ and

$$f(x,y)=\frac{xy}{x^2+y^2} \qquad \text{if } (x,y)\neq(0,0),$$

prove that $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exist at every point of R^2, although f is not continuous at $(0,0)$.

For $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$D_1 f(x,y) = \frac{\partial f(x,y)}{\partial x} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$$

$$D_2 f(x,y) = \frac{\partial f(x,y)}{\partial y} = \frac{x^3 - x y^2}{(x^2+y^2)^2}$$

Thus D_1f, D_2f exists for $\mathbb{R}^2 \setminus \{(0,0)\}$. Now we check $(0,0)$ using definition

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = 0$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = 0$$

So D_1f, D_2f exist everywhere in $\mathbb{R}^2$. To show f is discontinuous at $(0,0)$, we approach $(0,0)$ from the line $y=x$, but for all $\delta>0$, we select

$$d((x,x),(0,0)) < \delta \Rightarrow |f(x,x) - f(0,0)| = \frac{1}{2}$$

This shows that the δ-ε argument for continuity does not hold for $\varepsilon < \frac{1}{2}$.

2\. 7. Suppose that f is a real-valued function defined in an open set $E \subset R^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 9.21.

Since the partials are bounded, let

$$|D_i f(x)| < M_i \quad \forall x \in E, \quad i = 1, 2, \ldots, n$$

Define $M = \max_{i \in \{1,\ldots,n\}} M_i$, and choose $y \in E$ s.t. $d(x,y) < \frac{\varepsilon}{Mn}$ for $\varepsilon > 0$.

Following Rudin Thm. 9.21 proof, let $y - x = h = \sum_{i=1}^{n} h_i e_i$, $v_k = \sum_{j=1}^{k} h_j e_j$, $v_0 = (0, \ldots, 0)$. First, as in Rudin Thm. 9.21 proof, we get

$$f(x + v_i) - f(x + v_{i-1}) = h_i (D_i f(x + v_{i-1} + t_i h_i e_i)) \text{ for } t_i \in [0,1]$$

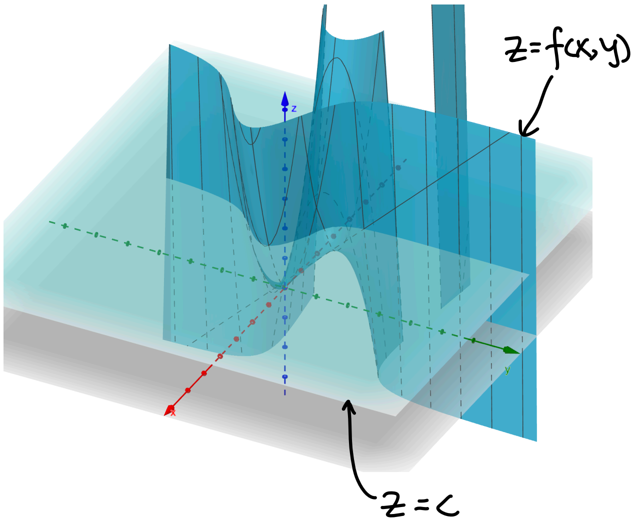
(this result can be established by MVT and chain rule as in Pugh pg. 284)

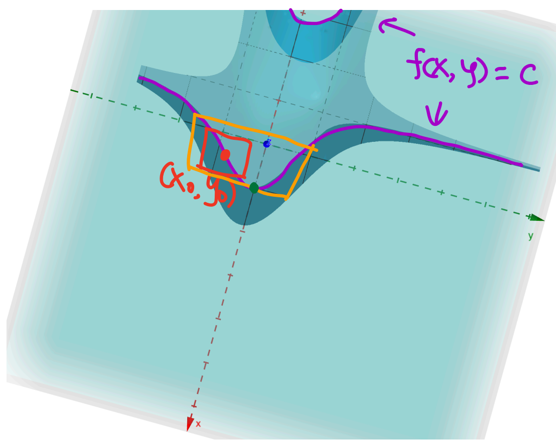
Now,

$$\begin{aligned}
|f(x) - f(y)| &= |f(x+h) - f(x)| \\
&= \left|\sum_{i=1}^{n} (f(x+v_i) - f(x + v_{i-1}))\right| \\
&\leq \sum_{i=1}^{n} |f(x+v_i) - f(x+v_{i-1})| \\
&= \sum_{i=1}^{n} |h_i (D_i f(x + v_{i-1} + t_i h_i e_i))| \\
&\leq \sum_{i=1}^{n} |h_i| |D_i f(x + v_{i-1} + t_i h_i e_i)| \\
&\leq M \sum_{i=1}^{n} |h_i| \\
&\leq M n |h| \\
&\leq M n |x, y| \\
&\leq \varepsilon
\end{aligned}$$

$\Rightarrow f$ is continuous in E

4. For graphical interpretation, I generated images from GeoZebra.





I believe this plot shows some nuances of the IFT. We see that around (x_0,y_0), the solution of $f(x,y)=c$, given a fixed x, is unique, but this neighbourhood has to be relatively small. On the right plot the neighbourhood in red works, but the orange one doesn't. Notice the local extremum (in green), where $\frac{\partial f}{\partial y}=0$ at that point, so the IFT does not work at that point.

Intuitive Interpretation:

We have a function $f(x,y)$. Fix c and $f(x_0,y_0)=c$. Then if $\frac{\partial f(x_0,y_0)}{\partial y}\neq 0$, for each x in a small neighbourhood of (x_0,y_0), $\exists$ a unique y s.t. $f(x,y)=c$. The condition on the partial ensures that something of this "$\cap$ or $\cup$" shape won't occur, so each x corresponds to one and only one y.