7.2.1

(v) Since a set of boxes of any volume can cover $\phi$,

$$m^*(\phi) = \inf\{\mathbb{R}^n \setminus (-\infty, 0)\} = 0$$

(vi) The volume of any box is nonnegative by definition.

So the infimum of the set of volumes of any collection of boxes is nonnegative.

(vii) Let $M = \{\sum_{j \in J_1} \mathrm{vol}(B_j) : (B_j)_{j \in J_1} \text{ covers } B\}$

$N = \{\sum_{j \in J_2} \mathrm{vol}(B_j) : (B_j)_{j \in J_2} \text{ covers } A\}$

Since $A \subseteq B$, $M \subseteq N$, therefore,

$\inf N \leq \inf M$

$\Rightarrow m^*(A) \leq m^*(B)$

(viii) We will use induction. Base case: $J = \{J_1, J_2\}$

$\forall \varepsilon > 0$, $\exists$ coverings $\{B_j\}_{j \in J_1}$, $\{B_j\}_{j \in J_2}$ of $A_1, A_2$ s.t.

$$\sum_{j \in J_1} |B_j| \leq m^*(A_1) + \frac{\varepsilon}{2}$$

$$\sum_{j \in J_2} |B_j| \leq m^*(A_2) + \frac{\varepsilon}{2}$$

$$\Rightarrow m^*(A_1) + m^*(A_2) + \varepsilon \geq \sum_{j \in J_1} |B_j| + \sum_{j \in J_2} |B_j|$$

Now, $\{B_j\}_{j \in (J_1 \cup J_2)}$ is a covering of $A_1 \cup A_2$. Then

$$\sum_{j \in (J_1 \cup J_2)} |B_j| \geq m^*(A_1 \cup A_2)$$

Since $\sum_{j \in J_1} |B_j| + \sum_{j \in J_2} |B_j| \geq \sum_{j \in (J_1 \cup J_2)} |B_j|$,

$$m^*(A_1) + m^*(A_2) + \varepsilon \geq m^*(A_1 \cup A_2)$$

Since $\varepsilon$ is arbitrary,

$$m^*(A_1) + m^*(A_2) + \varepsilon \geq m^*(A_1 \cup A_2)$$

Induction Step:

Let $\dot{J}_n = \dot{\bigcup}_{k=1}^{n} J_k$, assume $m^*(\dot{\bigcup}_{j \in \dot{J}_n} A_j) \leq \sum_{j \in \dot{J}_n} m^*(A_j)$.

N.T.S. true for $n+1$.

Let $A = \bigcup_{j \in \dot{J}_n} A_j$, then

$$\begin{aligned} m^*(\bigcup_{j \in \dot{J}_{n+1}} A_j) &= m^*(A \cup A_{n+1}) \\ &\leq m^*(A) + m^*(A_{n+1}) \\ &\leq \left[\sum_{j \in \dot{J}_n} m^*(A_j)\right] + m^*(A_{n+1}) \\ &= \sum_{j \in \dot{J}_{n+1}} m^*(A_j) \end{aligned}$$

(x) $\forall \varepsilon > 0$, $\exists$ coverings $\{B_m\}_{m \in J_i}$ of $A_i$, $i = 1, 2, \ldots$, s.t.

$$\sum_{m \in J_i} |B_m| \leq m^*(A_i) + \frac{\varepsilon}{2^i}$$

Then, $\sum_{i=1}^{\infty} \sum_{m \in J_i} |B_m| \leq \left[\sum_{i=1}^{\infty} m^*(A_i)\right] + \varepsilon$

Let $J = \dot{\bigcup}_{i=1}^{\infty} J_i$, then $\{B_m\}_{m \in J}$ is a covering of $\dot{\bigcup}_{i=1}^{\infty} A_i$. Therefore,

$$m^*(\dot{\bigcup}_{i=1}^{\infty} A_i) \leq \sum_{m \in J} |B_m|$$

Since $\sum_{m \in J} |B_m| \leq \sum_{i=1}^{\infty} \sum_{m \in J_i} |B_m|$, we have

$$m^*(\dot{\bigcup}_{i=1}^{\infty} A_i) \leq \left[\sum_{i=1}^{\infty} m^*(A_i)\right] + \varepsilon$$

Since $\varepsilon$ is arbitrary,

$$m^*(\dot{\bigcup}_{i=1}^{\infty} A_i) \leq \left[\sum_{i=1}^{\infty} m^*(A_i)\right]$$

(xiii) For every covering $\{B_j\}$ of $\Omega$,

$$\sum_j |B_j| = \sum_j |(B_j + x)|, \text{ and } \{B_j + x\} \text{ is a covering of } \Omega + x.$$

Therefore, the sets

$$\left\{ \sum_j |B_j| : \{B_j\} \text{ covers } \Omega \right\}$$

$$\left\{ \sum_j |B_j + x| : \{B_j\} \text{ covers } \Omega \right\}$$

are equivalent, so

$$m^*(\Omega) = m^*(\Omega + x)$$

## 7.2.2

$\forall \varepsilon > 0$, $\exists \{A_i\}, \{B_j\}$ that cover $A, B$, s.t.

$$\sum_i |A_i| \le m^*(A) + \varepsilon, \quad \sum_j |B_j| \le m^*(A) + \varepsilon$$

Then, for any $i, j$, $A_i \times B_j$ is also a box, so we have

$$|A_i| = m^*(A_i), \ |B_j| = m^*(B_j),$$

$$|A_i \times B_j| = m^*(A_i \times B_j)$$

$$|A_i \times B_j| = |A_i||B_j|$$

from Collary 7.2.7 and Definition 7.2.1.

Now, since $A \times B \subseteq \bigcup_{i,j} (A_i \times B_j)$,

$$\begin{aligned} m^*(A \times B) &\le m^*\Big(\bigcup_{i,j} (A_i \times B_j)\Big) \\ &\le \sum_{i,j}\sum m^*(A_i \times B_j) \\ &= \sum_{i,j}\sum |A_i||B_j| \\ &= \sum_i |A_i| \sum_j |B_j| \\ &\le (m^*(A) + \varepsilon)(m^*(B) + \varepsilon) \end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$m^*(A \times B) \le m^*(A)\, m^*(B)$$

## 7.2.3

(a) First we can rewrite

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} (A_j \setminus A_{j-1}), \text{ where } A_0 = \emptyset$$

So,

$$\begin{aligned}
m\left(\bigcup_{j=1}^{\infty} A_j\right) &= m\left(\bigcup_{j=1}^{\infty} (A_j \setminus A_{j-1})\right) \\
&= \sum_{j=1}^{\infty} [m(A_j) - m(A_{j-1})] \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} [m(A_j) - m(A_{j-1})] \\
&= \lim_{n\to\infty} m(A_n)
\end{aligned}$$

(b) $$\bigcap_{j=1}^{\infty} A_j = A_1 \setminus \bigcup_{j=1}^{\infty} (A_1 \setminus A_j)$$

$$\begin{aligned}
\Rightarrow m\left(\bigcap_{j=1}^{\infty} A_j\right) &= m(A) - m\left(\bigcup_{j=1}^{\infty} (A_1 \setminus A_j)\right) \\
&= m(A_1) - \lim_{n\to\infty} m(A_1 \setminus A_n) \quad \text{from (a), since } (A_1 \setminus A_j) \subseteq (A_1 \setminus A_{j+1}) \\
&= \lim_{n\to\infty} [m(A_1) - m(A_1 \setminus A_n)] \\
&= \lim_{n\to\infty} [m(A_1 \setminus (A_1 \setminus A_n))] \\
&= \lim_{n\to\infty} m(A_n)
\end{aligned}$$

7.2.4

First, we define a set of boxes translated from $(0,\frac{1}{q})^n$, let

$$A = \{(0,\tfrac{1}{q})^n + \tfrac{1}{q}(a_1, a_2, \dots, a_n) : a_1, \dots, a_n \in \{0,1,\dots,q-1\}\}$$

$\Rightarrow$ There are $q^n$ boxes in A, and $A \subset (0,1)^n$ since we exclude the boundaries of each box.

Therefore,

$$q^n\, m[(0,\tfrac{1}{q})^n] = m(A) \leq m[(0,1)^n] = 1$$

$$\Rightarrow m[(0,\tfrac{1}{q})^n] \leq q^{-n}$$

Second, let

$$B := \{[0,\tfrac{1}{q}]^n + \tfrac{1}{q}(b_1, b_2, \dots, b_n) : b_1, \dots, b_n \in \{0,1,\dots,q-1\}\}$$

be a set of translated boxes of $[0,\frac{1}{q}]^n$.

Now, B covers $[0,\frac{1}{q}]^n$ exactly, so

$$[0,1]^n = \bigcup_{i=1}^{q^n} B_i, \quad B_i \in B$$

Therefore,

$$1 = m([0,1]^n) = m(\bigcup_{i=1}^{q^n} B_i)$$

$$\leq \sum_{i=1}^{q^n} m(B_i)$$

$$= q^n([0,\tfrac{1}{q}]^n)$$

$$\Rightarrow m[0,\tfrac{1}{q}]^n \geq q^{-n}$$

Third, let $S_n$ be the number of sides (edges?) of $[0,\frac{1}{q}]^n$.

The boundary of $[0,\frac{1}{q}]^n$ can be represented as

$\bigcup_{j=1}^{S_n} (D_j \cup E_j)$, where

$$D_j = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j = 0,\ 0 \leq x_i \leq \tfrac{1}{q} \text{ for } i \neq j\}$$

$$E_j = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j = \tfrac{1}{q},\ 0 \leq x_i \leq \tfrac{1}{q} \text{ for } i \neq j\}$$

Then, $\forall \varepsilon > 0$, let $\delta = \left[\sum_{j=1}^{S_n}(|D_j| + |E_j|)\right]^{-1}\varepsilon$, $\exists$ coverings $C_j^D$, $C_j^E$ $D_j$, $E_j$, s.t.

$$C_j^D = \{(x_1, \dots, x_n) \in \mathbb{R} : -\tfrac{\delta}{2} < x_k < \tfrac{\delta}{2},\ 0 \leq x_l \leq \tfrac{1}{q} \text{ for } k \neq l\}$$

$$C_j^E = \{(x_1, \dots, x_n) \in \mathbb{R} : \tfrac{1}{q} - \tfrac{\delta}{2} < x_k < \tfrac{1}{q} + \tfrac{\delta}{2},\ 0 \leq x_l \leq \tfrac{1}{q} \text{ for } k \neq l\}$$

$$\Rightarrow |C_j^D| = \delta |D_j|,\quad |C_j^E| = \delta |E_i|$$

Therefore,

$$\begin{aligned}
m(\{[0,\tfrac{1}{q}]^n \setminus (0,\tfrac{1}{q})^n\}) &\leq m\left(\bigcup_{j=1}^{S_n} (D_j \cup E_j)\right) \\
&\leq \sum_{j=1}^{S_n} m(D_j \cup E_j) \\
&\leq \sum_{j=1}^{S_n} (m(D_j) + m(E_j)) \\
&\leq \sum_{j=1}^{S_n} (|C_j^D| + |C_j^E|) \\
&= \sum_{j=1}^{\infty} \delta(|D_j| + |E_j|) \\
&= \varepsilon
\end{aligned}$$

Since $(0,\frac{1}{q})^n$ and $\{[0,\frac{1}{q}]^n \setminus (0,\frac{1}{q})^n\}$ are disjoint,

$$\begin{aligned}
q^{-n} &\leq m([0,\tfrac{1}{q}]^n) \\
&= m[(0,\tfrac{1}{q})^n] + m[\{[0,\tfrac{1}{q}]^n \setminus (0,\tfrac{1}{q})^n\}] \\
&\leq m[(0,\tfrac{1}{q})^n] + \varepsilon \\
&\leq q^{-n} + \varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$,

$$m[(0,\tfrac{1}{q})^n] = q^{-n}$$

Also,

$$\begin{aligned}
q^{-n} &\geq m[(0,\tfrac{1}{q})^n] \\
&= m([0,\tfrac{1}{q}]^n) - m(\{[0,\tfrac{1}{q}]^n \setminus (0,\tfrac{1}{q})^n\})
\end{aligned}$$

$$\geq m([0,\tfrac{1}{q}]^n) - \varepsilon$$

$$\geq q^{-n} - \varepsilon$$

Letting $\varepsilon \to 0$,

$$m([0,\tfrac{1}{q}]^n) = q^{-n}$$

7.4.1

If (1) $A \cap (0,\infty) = \phi$ or (2) $A \cap (-\infty, 0) = \phi$, then

(1) $m^*(A) = 0 + m^*(A \setminus (0,\infty))$

$= m^*(A)$

(2) $m^*(A) = m^*(A \cap (0,\infty)) + 0$

$= m^*(A)$

If neither (1) or (2) is true, let $A = (a,b)$. By Collary 7.2.7,

$m^*[A \cap (0,\infty)] = m^*[(0,b)] = b$

$m^*[A \setminus (0,\infty)] = m^*[(a,0]] = -a$

Therefore,

$m^*[A \cap (0,\infty)] + m^*[A \setminus (0,\infty)] = b - a = m^*(A)$

7.4.2

Let $A = \prod_{i=1}^{n} (a_i, b_i)$ as defined by definition 7.2.1.

Similar to Exercise 7.4.1, if

(1) $(a_n, b_n) \cap (0, \infty) = \emptyset \Rightarrow A \cap E = \emptyset,\ A \setminus E = A$

$$\Rightarrow m^*(A) = 0 + m^*(A) = m^*(A)$$

(2) $(a_n, b_n) \cap (0, \infty) = (a_n, b_n) \Rightarrow A \cap E = A,\ A \setminus E = \emptyset$

$$\Rightarrow m^*(A) = m^*(A \cap E) + 0 = m^*(A)$$

Now, if $a_n < 0$, $b_n > 0$, first,

$$m^*(A) = \prod_{i=1}^{n} (b_i - a_i) = \left[\prod_{i=1}^{n-1} (b_i - a_i)\right](b_n - a_n)$$

Define $B = A \cap E$, $B^c = A \setminus E$, then $B, B^c$ are boxes as well. So

$$m^*(B) = \left[\prod_{i=1}^{n-1} (b_i - a_i)\right] b_n$$

$$m^*(B^c) = \left[\prod_{i=1}^{n-1} (b_i - a_i)\right](-a_n)$$

$$\Rightarrow m^*(B) + m^*(B^c) = \left[\prod_{i=1}^{n-1} (b_i - a_i)\right](b_n - a_n)$$

$$= \prod_{i=1}^{n} (b_i - a_i)$$

$$= m^*(A)$$

7.4.3

Let $D$ be the half space defined in the Lemma

$\forall \varepsilon > 0$, $\exists B := \{B_j\}$ a covering of $A$ s.t.

$$\sum_j |B_j| \le m^*(A) + \varepsilon$$

By Exercise 7.4.2,

$$m^*(B_j) = m^*(B_j \cap D) + m^*(B_j \setminus D)$$

Since $A \subseteq B$,

$$A \cap D \subseteq B \cap D, \quad A \setminus D \subseteq B \setminus D$$

$$\Rightarrow m^*(A \cap D) \le m^*(B \cap D)$$

$$m^*(A \setminus D) \le m^*(B \setminus D)$$

Now, since $A = (A \cap D) \cup (A \setminus D)$,

$$\begin{aligned} m^*(A) &\le m^*(A \cap D) + m^*(A \setminus D) \\ &\le m^*(B \cap D) + m^*(B \setminus D) \\ &= m^*\Big(\bigcup_j (B_j \cap D)\Big) + m^*\Big(\bigcup_j (B_j \setminus D)\Big) \\ &\le \sum_j \big[m^*(B_j \cap D) + m^*(B_j \setminus D)\big] \\ &= \sum_j m^*(B_j) \\ &\le \sum_j |B_j| \\ &\le m^*(A) + \varepsilon \end{aligned}$$

Since $\varepsilon$ is arbitrary,

$$m^*(A) = m^*(A \cap D) + m^*(A \setminus D)$$

## 7.4.4

(a) For any $A \subset \mathbb{R}^n$,

$$A \cap (\mathbb{R}^n \setminus E) = A \setminus E$$

$$A \setminus (\mathbb{R}^n \setminus E) = A \cap E$$

Therefore,

$$\begin{aligned} m^*(A) &= m^*(A \cap (\mathbb{R}^n \setminus E)) + m^*(A \setminus (\mathbb{R}^n \setminus E)) \\ &= m^*(A \setminus E) + m^*(A \cap E) \\ &= m^*(A) \text{ since } A \text{ is measurable} \end{aligned}$$

(b) For any $A \subset \mathbb{R}^n$, since $E$ is measurable,

$$m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x) \setminus E)$$

And,

$$m^*(A-x) = m^*(A)$$

$$m^*((A-x) \cap E) = m^*(A \cap (E+x))$$

$$m^*((A-x) \setminus E) = m^*(A \setminus (E+x))$$

$\Rightarrow m^*(A) = m^*(A \cap (E+x)) + m^*(A \setminus (E+x))$

$\Rightarrow$ $E+x$ is measurable

$\Rightarrow$ $m(E) = m(E+x)$ follows from (xiii) of 7.1

(c) Let $E_i^c = \mathbb{R}^n \setminus E_i$, $i \in \{1, 2\}$

$$A_1 = A \cap E_1 \cap E_2$$
$$A_2 = A \cap E_1 \cap E_2^c$$
$$A_3 = A \cap E_1^c \cap E_2^c$$
$$A_4 = A \cap E_1^c \cap E_2$$

W.T.S.: $\forall A \subset \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \setminus (E_1 \cap E_2))$$

Now,

$$A \cap (E_1 \cap E_2) = A_1$$

$$A \setminus (E_1 \cap E_2) = A_2 \cap A_3 \cap A_4$$

So we W.T.S.

$$m^*(A) = m^*(A_1) + m^*(A_2 \cap A_3 \cap A_4)$$

First,

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \quad \text{since } E_1 \text{ measurable}$$

$$= m^*(A_1 \cup A_2) + m^*(A_3 \cup A_4) \qquad (1)$$

Then, since $E_2$ is measurable,

$$m^*(A_1 \cup A_2) = m^*((A_1 \cup A_2) \cap E_2) + m^*((A_1 \cup A_2) \setminus E_2)$$

$$= m^*(A_1) + m^*(A_2) \qquad (2)$$

$$m^*(A_3 \cup A_4) = m^*((A_3 \cup A_4) \cap E_2) + m^*((A_3 \cup A_4) \setminus E_2)$$

$$= m^*(A_4) + m^*(A_3) \qquad (3)$$

Then, we can show

$$m^*(A_2 \cup A_3 \cup A_4) = m^*((A_2 \cup A_3 \cup A_4) \cap E_1) + m^*((A_2 \cup A_3 \cup A_4) E_1)$$

$$= m^*(A_2) + m^*(A_3 + A_4)$$

$$= m^*(A_2) + m^*(A_3) + m^*(A_4) \qquad (4)$$

From all above,

$$m(A) = m^*(A_1 \cup A_2) + m^*(A_3 \cup A_4) \quad \text{by (1)}$$

$$= m^*(A_1) + m^*(A_2) + m^*(A_3) + m^*(A_4) \quad \text{by (2), (3), call this (5)}$$

$$= m^*(A_1) + m^*(A_2 \cup A_3 \cup A_4) \quad \text{by (4)}$$

$\Rightarrow E_1 \cap E_2$ measurable

Then, W.T.S. : $\forall A \subset \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \setminus (E_1 \cup E_2))$$

First,

$$A \cap (E_1 \cup E_2) = A_1 \cup A_2 \cup A_4$$

$$A \setminus (E_1 \cup E_2) = A_3$$

So we W.T.S. :

$$m^*(A) = m^*(A_1 \cup A_2 \cup A_4) + m^*(A_3)$$

Since $E_1$ is measurable,

$$\begin{aligned} m^*(A_1 \cup A_2 \cup A_4) &= m^*(A_1 \cup A_2 \cup A_4) \cap E_1) + m^*((A_1 \cup A_2 \cup A_4) \setminus E_1) \\ &= m^*(A_1 \cup A_2) + m^*(A_4) \\ &= m^*(A_1) + m^*(A_2) + m^*(A_4) \quad \text{by (2)} \end{aligned}$$

Therefore,

$$\begin{aligned} m^*(A) &= m^*(A) + m^*(A_2) + m^*(A_3) + m^*(A_4) \quad \text{by (5)} \\ &= m^*(A_1 \cup A_2 \cup A_4) + m^*(A_3) \end{aligned}$$

(d) We will use induction. <u>Base Case</u> : $N=2$

This is shown by (c) :

$E_1 \cap E_2$, $E_1 \cup E_2$ measurable

<u>Induction Step</u> :

Assume $E_a := \bigcup_{j=1}^{N-1} E_j$ and $E_b := \bigcap_{j=1}^{N-1} E_j$ are measurable.

Then, from (c)

$E_a \cup E_N = \bigcup_{j=1}^{N} E_j$ and

$E_b \cap E_N = \bigcap_{j=1}^{N} E_j$ are measurable.

(e) Let $B = \prod_{i=1}^{n} (a_i, b_i) = \{(x_1, \dots, x_n) : a_i < x_i < b_i,\ i \in \{1, \dots, n\}\}$

Then we can express $B$ as $A_1 \cap A_2$, where

$$A_1 = \{(x_1, \dots, x_n) : a_i < x_i < \infty,\ i \in \{1, \dots, n\}\}$$

$$A_2 = \{(x_1, \dots, x_n) : -\infty < x_i < b_i,\ i \in \{1, \dots, n\}\}$$

But $A_1, A_2$ are translated from the half spaces

$$E_1 = \{(x_1, \dots, x_n) : x_n > 0\}$$

$$E_2 = \{(x_1, \dots, x_n) : x_n < 0\}$$

Namely,

$A_1 = x_1 + E_1$, $A_2 = x_2 + E_2$, namely

$$x_1 = (a_1, a_2, \dots, a_n)$$

$$x_2 = (b_1, b_2, \dots, b_n)$$

Therefore,

$E_1, E_2$ are measurable (Lemma 7.4.2)

$\Rightarrow A_1, A_2$ measurable (part (b))

$\Rightarrow B = A_1 \cap A_2$ measurable (part (c))

(f) To show $E$ measurable, we show

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

Let $m^*(E) = 0$, then $\forall A \subset \mathbb{R}^n$,

$$m^*(A \cap E) \le m^*(E) = 0$$

$$\Rightarrow m^*(A \cap E) = 0$$

Since $(A \setminus E) \subset A$,

$$m^*(A) \geq m^*(A \setminus E) = m^*(A \setminus E) + m^*(A \cap E) \qquad (1)$$

Then, since $((A \cap E) \cup (A \setminus E)) = A$,

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E) \tag{2}$$

By (1) and (2)

$$m(A) = m^*(A \cap E) + m^*(A \setminus E)$$