$\boxed{\text{Thm}}$ If f, g are simple fxns. then

$$\int f+g = \int f + \int g$$

$\boxed{\text{Pf}}$ $f = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}(x)$, $c_i > 0$, E_i disj.

$g = \sum_{i=1}^{M} d_i \mathbb{1}_{F_i}(x)$.

Define $E_0 = \mathbb{R}^n \setminus E_1 \cup \cdots \cup E_N$, $c_0 = 0$

$F_0 = \mathbb{R}^n \setminus F_1 \cup \cdots \cup F_M$, $d_0 = 0$

Then $\mathbb{R}^N = \bigcup_{i=0}^{N} \bigcup_{j=0}^{M} (E_i \cap F_j)$, $f+g = \sum_{i,j=0}^{N,M} (c_i + d_j) \mathbb{1}_{E_i \cup F_i}$

$$\Rightarrow \int f+g = \sum_{i=0}^{N} \sum_{j=0}^{M} (c_i + d_j) m(E_i \cap F_i)$$

$$= \sum_{i,j} c_i m(E_i \cap F_j) + \sum_{i,j} d_j m(E_i \cap F_j)$$

$$= \sum_i c_i m(E_i) + \sum_j d_j m(F_j)$$

$$= \int f + \int g$$

Tao 8.2 Integration on Non-Neg. Meas. Fxn.

- Let $f \geq 0$ be measurable $\Omega \to [0, \infty]$ ($\subset \mathbb{R}$?, extension), then

$$\int f = \sup \{ \int s \mid s \text{ simple}, s \geq 0, s \leq f \}$$

* Remark

For a seq. s_n where $0 \leq s_1 \leq s_2 \leq \cdots$ and $\sup(s_n(x)) = f(x)$ (pt wise conv.)

- $\int f = \sup \int s_n$ by MCT
- Can't prove just by def. ($\int s_n$ may not achieve sup)

<u>Prop 8.2.6</u> If $f, g : \Omega \to [0,\infty]$ meas., then

(1) $\int f \geq 0$ and $\int f = 0$ iff $f(x) = 0$ a.e. $\left(\begin{array}{l}\text{i.e. } \exists Z \subset \Omega \text{ null set} \\ \text{s.t. } f|_{\Omega \setminus Z} = 0\end{array}\right)$

(2) $\forall c > 0$, $\int c \cdot f = c \cdot \int f$

(3) $f \leq g \Rightarrow \int f \leq \int g$

(4) If $f = g$ a.e. then $\int f = \int g$

$\boxed{\text{pf}}$ (1) If $f(x) \geq 0$, $f(x) = 0$ a.e., then any simple fxn.

$0 \leq s(x) \leq f(x)$, $s = \sum c_i \mathbb{1}_{E_i}(x)$, $c_i > 0$

$\Rightarrow \int s = \sum c_i \cdot m(E_i) = \sum c_i \cdot 0 = 0$

null since $E_i \subset Z = \{x \mid f(x) > 0\}$

(4) Let $Z = \{x : f(x) \neq g(x)\}$, $Z^c = \Omega \setminus Z$

$\forall s$ simple, $\int s = \int s \cdot \mathbb{1}_{Z^c}$

$\Rightarrow \int f = \int f \cdot \mathbb{1}_{Z^c} = \int g \cdot \mathbb{1}_{Z^c} = \int g$

$\boxed{\text{thm}}$ Given measurable fxns.,

$f_n : \Omega \to [0,\infty]$, $0 \leq f_1(x) \leq f_2(x) \leq \cdots$, then

(1) $0 \leq \int f_1 \leq \int f_2 \leq \cdots$

(2) $\int \sup f_n(x) = \sup \int f_n$

$\boxed{\text{pf}}$ Since $\sup f_n(x) = f(x) \geq f_n(x)$ $\forall n$,

$\int f \geq \int f_n$ $\forall n$ $\Rightarrow \int f \geq \sup \int f_n$

W.T.S. :

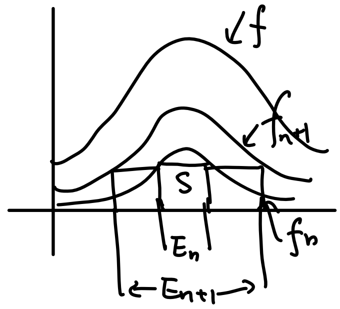
$\int f \leq \sup_n \int f_n \Leftrightarrow \forall s$ simple, $0 \leq s \leq f$, $\int s \leq \sup_n \int f_n$

$\Leftrightarrow \forall\, 1 > \varepsilon > 0$, $\forall s$ simple, $0 \leq s \leq f$

$(1-\varepsilon) \int s \leq \sup \int f_n$

Since $s(x) \le \sup f_n$, define

$$E_n = \{x \in \Omega \mid f_n(x) \ge (1-\varepsilon) s(x)\}$$



then

$$E_n \subset E_{n+1} \subset \cdots, \text{ and } \bigcup E_n = \Omega$$

$$\Rightarrow \int_{E_n} (1-\varepsilon) s_n \le \int_{E_n} f_n \le \int_\Omega f_n \le \sup_n \int_\Omega f_n$$

$$\Rightarrow \lim_{n\to\infty} \int_{E_n} (1-\varepsilon) s = \int_\Omega (1-\varepsilon)\, s \Rightarrow (1-\varepsilon)\int_\Omega s_n = \int_\Omega (1-\varepsilon)\, s_n \le \sup_n \int_\Omega f_n$$

A.V. $s = \sum c_i \mathbb{1}_{E_i}(x)$

$$\int_{E_n} s = \sum c_i \cdot m(F_i \cap E_n)$$

$$\Rightarrow E_n \nearrow \Omega \Rightarrow E_n \cap F \nearrow \Omega \cap F = F \Rightarrow m(E_n \cap F) \nearrow m(F)$$

$$\Rightarrow \lim \int_{E_n} s = \sum c_i m(F_i) = \int_\Omega s$$

Prop: If $f, g : \Omega \to [0,\infty]$ meas., then $\int f+g = \int f + \int g$

$\boxed{\text{Pf}}$ Let s_n be the sequence of simple fxns. $s_n \nearrow f$

t_n —— " —— $t_n \nearrow g$ $\Big\} \Rightarrow s_n + t_n \nearrow f+g$

By MCT, $\lim \int s_n = \int f$, $\lim \int t_n = \int g$. $\lim \int s_n + t_n = \int f+g$.

Since $\int s_n + t_n = \int s_n + \int t_n$,

$$\lim \int s_n + t_n = \lim \int s_n + \lim \int t_n$$

$\boxed{\text{Corollary}}$ If $g_1, g_2, \dots$ are non-neg. meas. fxns, then

$$\int_\Omega \sum_{n=1}^\infty g_n(x) = \sum_{n=1}^\infty \int_\Omega g_n(x)$$

$\boxed{\text{Pf}}$ Define $f_n = \sum_{m=1}^n g_m(x)$, then

$$\int \sup f_n \overset{MCT}{=} \sup \int f_n = \sup_N \int \sum_{n=1}^N g_n = \sup_N \sum_{i=1}^N \int g_n$$

Prop: If $f: \Omega \to [0,\infty]$ meas., $\int f < \infty$, then

? $f(x)$ is finite a.e. $\left(\begin{array}{l} \text{i.e. } \exists Z \text{ null set, that } f|_{\Omega \setminus Z} \text{ is finite} \\ \text{i.e. } f^{-1}(\infty) \text{ is a null set} \end{array}\right)$

$\boxed{\text{Corollary}}$ (Borel - Cantelli) If $\Omega_1, \Omega_2, \ldots$ are meas. set s.t. $\sum_{i=1}^{\infty} m(\Omega_i) < \infty$, then $\{x \mid x \text{ belong to infinitely many } \Omega_n\}$ is a null set.

$\boxed{\text{pf}}$ Since $m(\Omega_i) = \int_{\Omega_i} 1 = \int_\Omega \mathbb{1}_{\Omega_i}$,

$$\sum_i^{\infty} m(\Omega_i) = \sum_{i=1}^{\infty} \int_\Omega \mathbb{1}_{\Omega_i} = \int \underbrace{\sum_{i=1}^{\infty} \mathbb{1}_{\Omega_i}(x)}_{f(x)} < \infty$$

$\Rightarrow f^{-1}(\infty)$ is a null set.