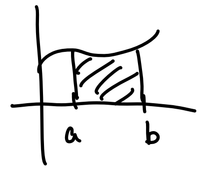
- **Riemann integral over $\mathbb{R}$**



$\int_a^b f(x)dx$ = area under the curve

↳ piecewise con't fxns. are Riemann integrable (including piecewise cons. fxns.)

* Shortcomings

- Underlying space : $\mathbb{R}$, $\mathbb{R}^n$
  - → Domain of integration only $[a,b]$ bounded
- Only bounded fxns.
- If $f_n \to f$ pt.wise, and f_n Riemann integrable, it is <u>not true</u> that f is Riemann integrable

- **Lebesgue Integral** : $\int_\Omega f\,dx$, where $\Omega \subset \mathbb{R}^n$ is some subset, $dx = dx_1 \cdots dx_n$
  - Ω : Lebesgue measurable sets
  - f : Lebesgue integrable fxns.

- **Lebesgue measure** : $m(\Omega) = \int_\Omega 1\,dx$
  - e.g. $\int_a^b 1\,dx = b-a = |[a,b]|$
  - $\Omega \subset \mathbb{R} \to m(\Omega)$ = length of Ω

    $\Omega \subset \mathbb{R}^2 \to m(\Omega)$ = area — " —

    $\Omega \subset \mathbb{R}^3 \to m(\Omega)$ = volume — " —
  - Need to "<u>consistently</u>" define measure for any subset of $\mathbb{R}^n$

    ↳ (1) Monotone : If $A \subset B \subset \mathbb{R}^n$, then $m(A) \le m(B)$

    (2) Additivity : If $A \cap B = \phi$, then $m(A \cup B) = m(A) + m(B)$

    (3) Translation invariance : $\forall x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$,

$m(E) = m(x+E) := \{x+a \mid a \in E\}$

↳ e.g. $3 + [1,2] = [4,5]$

* Not possible to define such a measure on all subsets on $\mathbb{R}^n$

↳ Solution: Restrict the class of subsets (measurable) in $\mathbb{R}^n$, to which we assign a measure.

→ Desired properties:

* Notation: If S is a set, 2^S denote the set of subsets in S

Let M_n denote the set of measurable subsets in $\mathbb{R}^n$. We want

(4) If $U \subset \mathbb{R}^n$ is open, then $U \in M_n$ (U is measurable)

(5) If $U \in M_n$, then $U^c = \mathbb{R}^n \setminus U$ is also in M_n

(6) If $U, V \in M_n$, then $U \cap V$ and $U \cup V$ are measurable

* (5)+(6) → M_n is a Boolean algebra

↳ 3 operations: $\underset{\text{NOT}}{(\ )^c}$, $\underset{\text{AND}}{(\ )\cap(\ )}$, $\underset{\text{OR}}{(\ )\cup(\ )}$

(7) We want M_n to be a σ-algebra (a.k.a. countable)

↳ If $U_1, U_2, \ldots$ is a sequence of measurable set, then $\bigcup_n U_n$ is measurable, $\bigcap_n U_n$ is measurable

- Axioms for Lebesgue measure (read Tao)

$\boxed{\text{Thm.}}$ $\exists$ a definition of M_n and Lebesgue measure, satisfying all the axioms.

↳ $m_n : M_n \to \mathbb{R}_{\geq 0}$

- Outer measure: $\forall E \subset \mathbb{R}^n$,

vol

$\{B_i\}$ is an open cover of E by boxes

↳ $m^*(E) := \inf\{\sum_{i=1}^{\infty} |B_i|, \bigcup_{i=1}^{\infty} B_i \supset E, B_i \subset \mathbb{R}^n \text{ are open boxes}\}$

* open boxes: $\mathbb{R}$: (a,b)

$\mathbb{R}^n$: $(a_1,b_1) \times (a_2,b_2) \times \cdots \times (a_n,b_n)$

- outer measure is defined for ALL subsets in $\mathbb{R}$

Lemma 7.2.5, 7.2.6

↳ $m^*(\text{box}) = \text{vol}(\text{box})$