$\boxed{\text{Lemma}}$ 7.2.5

Lemma 7.2.5 (Properties of outer measure). *Outer measure has the following six properties:*

(v) *(Empty set) The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.*

← no box is needed to cover

(vi) *(Positivity) We have $0 \le m^*(\Omega) \le +\infty$ for every measurable set Ω.*

↖ inf over nonnegative #

(vii) *(Monotonicity) If $A \subseteq B \subseteq \mathbf{R}^n$, then $m^*(A) \le m^*(B)$.*

$\boxed{\text{Pf}}$ For any open cover $\{B_i\}$ on B, it is also an open cover on A.

And if $M, N \subset \mathbb{R}$, $M \supset N$, then

$\inf M \le \inf N$

Thus $m^*(A) \le m^*(B)$

(viii) *(Finite sub-additivity) If $(A_j)_{j\in J}$ are a finite collection of subsets of $\mathbf{R}^n$, then $m^*(\bigcup_{j\in J} A_j) \le \sum_{j\in J} m^*(A_j)$.*

$\boxed{\text{Pf}}$ Show: $m^*(A \cup B) \le m^*(A) + m^*(B)$

Idea: Try proving $m^*(A) + m^*(B) \geq$ Area (covering of A) + Area (covering of B) $- \varepsilon$

$\Rightarrow$ Total area $\geq m^*(A \cup B)$

$\Rightarrow \forall \varepsilon > 0$, $m^*(A) + m^*(B) \geq m^*(A \cup B) - \varepsilon$

$\Rightarrow m^*(A) + m^*(B) \geq m^*(A \cup B)$

By definition, $\forall \varepsilon > 0$, $\exists$ covering $\{B_i\}$ s.t.

$$\sum_i |B_i| \le m^*(A) + \varepsilon/2$$

Same for B, then union the 2 countable covers to get a cover of $A \cup B$.

(x) *(Countable sub-additivity) If $(A_j)_{j\in J}$ are a countable collection of subsets of $\mathbf{R}^n$, then $m^*(\bigcup_{j\in J} A_j) \le \sum_{j\in J} m^*(A_j)$.*

$\boxed{\text{Pf}}$ W.T.S. : $\forall \varepsilon > 0$, $\exists$ a collection of open covers, $\{B_i^{(j)}\}$ for A_j s.t.

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_j m^*(A_j) + \varepsilon = \sum_{j=1}^{\infty}\left(m^*(A_j) + \frac{\varepsilon}{2^j}\right)$$

Can find open cover $\{B_i^{(j)}\}$ for A_j s.t.

$$m^*(A_j) + \frac{\varepsilon}{2^j} \geq \sum_{i=1}^{\infty} |B_i^{(j)}|$$

And

$$\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} |B_i^{(j)}|\right) \geq m^*\left(\bigcup_{j=1}^{\infty} A_j\right)$$

(xiii) *(Translation invariance) If Ω is a subset of $\mathbf{R}^n$, and $x \in \mathbf{R}^n$, then $m^*(x+\Omega) = m^*(\Omega)$.* ← translate the cover

Proposition 7.2.6 (Outer measure of closed box). *For any closed box*

$$B = \prod_{i=1}^{n} [a_i, b_i] := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_i \in [a_i, b_i] \text{ for all } 1 \le i \le n\},$$

we have

$$m^*(B) = \prod_{i=1}^{n} (b_i - a_i).$$

Recall: • compact set in $\mathbb{R}^n \iff$ closed & bounded

• Riemann integral:

- $\mathrm{vol}([a,b]) = b-a = \int_a^b 1\, dx = \int_{\mathbb{R}} \mathbb{1}_{[a,b]}(x)\, dx$, where

$$\mathbb{1}_{[a,b]}(x) = \begin{cases} 1, & x \in [a,b] \\ 0 & \text{o/w} \end{cases}$$

- n-dim: $\mathrm{vol}(\underbrace{[a_1,b_1] \times \cdots \times [a_n,b_n]}_{B}) = \int_{\mathbb{R}^n} \mathbb{1}_B(x)\, dx$

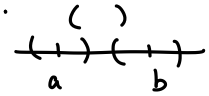
*same for open boxes

$\boxed{\text{Pf}}$ obs: can choose an open box, slightly larger than B to cover B, thus

$$m^*(B) \le \mathrm{vol}(B) + \varepsilon \quad \forall \varepsilon > 0 \implies m^*(B) \le \mathrm{vol}(B)$$

($n=1$) case: b/c $B = [a,b]$ is compact, hence any open cover of B can be reduced to a finite subcover

Let $\{B_i\}_{i=1}^N$ be a finite open cover of B e.g.



W.t.s.:

$$\sum_{i=1}^N |B_i| \ge \mathrm{vol}(B)$$

Let $f_i(x) = \mathbb{1}_{B_i}(x)$, then

?

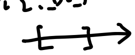
$$\sum_{i=1}^N |B_i| = \sum_{i=1}^N \left(\int_{\mathbb{R}} f_i\, dx\right) = \int_{\mathbb{R}} \overbrace{\sum_{i=1}^N f_i}^{f(x)}\, dx \ge \int \mathbb{1}_B(x)\, dx = \mathrm{vol}(B)$$

Claim: $f(x) \ge \mathbb{1}_{B_i}$

Indeed, $B \subset \bigcup_{i=1}^N B_i$, thus $\mathbb{1}_B \le \sum \mathbb{1}_{B_i}$

($n=2$ case): w.t.s. given any finite cover $\{B_i\}_{i=1}^N$ of B that $\sum_{i=1}^N |B_i| > |B_i|$

Again $|B_i| = \int_{\mathbb{R}^2} \mathbb{1}_{B_i}(x_1, x_2)\, dx_1\, dx_2$, integrate along x_2 w.r.t. $\{B_i\}$



$$= \int_{\mathbb{R}} w_i \cdot \mathbb{1}_{B_{i,1}}(x)\, dx_1$$

$$\sum_{i=1}^{N} \int \mathbb{1}_{B_i}(x)\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}^2} \sum_{i=1}^{N} \mathbb{1}_{B_i}(x)\, dx_1\, dx_2$$

? $$= \int_{\mathbb{R}} \Big(\underbrace{\int \sum_i \mathbb{1}_{B_i}(x_1, x_2)\, dx_2}_{f(x_1)} \Big)\, dx_1$$

claim: $f(x_1) \geq \mathbb{1}_{[a_1,b_1]}(x_1) \cdot \underbrace{|b_2 - b_1|}_{\text{height of } B}$ $\Longleftrightarrow$ $\left\{ \begin{array}{l} \text{if } x_1 \in [a_1,b_1],\ f(x_1) \geq |b_2 - b_1| \\ \text{if } x_2 \notin [a_2,b_2],\ f(x_1) \geq 0 \text{ is true} \end{array} \right\}$

follows by induction hypothesis ($n=1$ case) applied to the line w/ the given x_1.

$$\geq \int_{\mathbb{R}} \mathbb{1}_{[a_1,b_1]}(x_1)(b_2 - a_2)\, dx_1 = (b_2 - a_2)(b_1 - a_1)$$
$$= \mathrm{vol}(B)$$

General n: by induction

Pugh: divide B into grids of smaller boxes so that ea. small box is contained in some B_i.

- Then $\mathrm{vol}(B) = \sum \mathrm{vol}(\text{grid of small boxes})$
  $\leq \sum \mathrm{vol}(\text{open cover } B_i)$

$\boxed{\text{Corollary}}$ Outer measure of any box (open, closed, half & half $= \mathrm{vol}(\text{box})$
e.g. $[a_1, b_1) \times (a_2, b_2]$

We then have

- $m^*(\mathbb{N}) = 0$ (by countable sub-additivity)

  $$m^*(\mathbb{N}) \leq \sum_{i=0}^{\infty} m^*(\{i\}) = \sum_{i=1}^{\infty} 0 = 0$$

- Similarly, $m^*(\mathbb{Q}) = 0$
- $m_1^*(\mathbb{R}) = \infty$ by monotonicity

  $\hookrightarrow m_1^*((-R, R)) = 2R \Rightarrow m_1^*(\mathbb{R}) \geq 2R \quad \forall R > 0$

$\Rightarrow m_1^*(\mathbb{R}) = \infty$

<u>Skip</u> Tao 7.3

Idea: • construct a "weird" subset $E \subset [0,1]$

• $[0,2] \supset \bigsqcup_{q\in[0,1]\cap\mathbb{Q}} q+E \supset [0,1]$ ← disjoint union

• trouble: additivity would fail

$$m^*\Big(\bigsqcup_{q\in[-1,1]\cap\mathbb{Q}} q+E\Big) = \sum m^*(q+E) = \sum m^*(E) = \begin{matrix} 0\,? \\ \infty\,? \end{matrix}$$

$$\Rightarrow \underbrace{m^*([0,1])}_{=\,1} \le m^*\Big(\bigsqcup_{q} q+E\Big) \le m^*([-1,2]) = 3$$

$\{W_n \le k\}$ exclude $\{W_n \le k-1\}$ $=$ $\{W_n = k\}$

$$\mathbb{P}(W_n \le k) - \mathbb{P}(W_n \le k-1) = \mathbb{P}(W_n = k)$$