Recall: • $m^*(A) = \inf\{\sum_i |B_i|, \{B_i\}$ is countable collection of open boxes covering $A\}$

• Measurable Set: $E \subset \mathbb{R}^n$ is called measurable iff $\forall A \subset \mathbb{R}^n$

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

↖ Catheodory Property

Lemma 7.4.2 (Half-spaces are measurable). *The half-space*

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$$

is measurable.

$\boxed{\text{pf}}$ $(n=1)$: WTS: $\forall A \subset \mathbb{R}$,

$$m^*(A) = m^*(A_+) + m^*(A_-),$$

where $A_+ = A \cap (0,\infty)$, $A_- = A \cap (-\infty, 0]$

(1) $A = A_+ \overset{\text{disjoint union}}{\sqcup} A_-$

$\Rightarrow m^*(A) \leq m^*(A_+) + m^*(A_-)$ by sub-additivity

(2) To show $m^*(A) \geq m^*(A_+) + m^*(A_-)$, it suffice to show

$\forall \varepsilon > 0$,

$$m^*(A) + \varepsilon \geq m^*(A_+) + m^*(A_-)$$

Consider an open cover of A by $\{B_j\}$ open boxes s.t.

$$\sum_{j=1}^{\infty} |B_j| \leq m^*(A) + \frac{\varepsilon}{2}$$

Define $B_j^+ = B_j \cap (0,\infty)$, $B_j^- = B_j \cap (-\infty, \varepsilon/2^{j+1})$

then • $B_j = B_j^- \cup B_j^+$, and $|B_j| + \frac{\varepsilon}{2^{j+1}} \geq |B_j^+| + |B_j^-| \geq |B_j|$

• $\bigcup B_j^+ \supset A_+$, $\bigcup B_j^- \supset A_-$

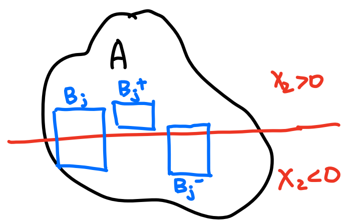
Thus $m^*(A_+) + m^*(A_-) \leq \sum |B_j^+| + \sum |B_j^-| \leq \sum_{j+1}^{\infty} (|B_j| + \frac{\varepsilon}{2^{j+1}})$

$$\leq (\sum |B_j|) + \frac{\varepsilon}{2} \leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = m^*(A) + \varepsilon$$

$(n=2)$

$A_+ = A \cap \{x_2 > 0\}$

$A_- = A \cap \{x_2 \leq 0\}$



Want: $B_j^- \supset B_j \cap \{x_2 \leq 0\}$

N.T.S. : $m^*(A) + \varepsilon \geq m^*(A_+) + m^*(A_-)$

* one can do the same : ① get $\{B_j\}$, cover of A w/ $m^*(A) + \frac{\varepsilon}{2} \geq \sum |B_j|$

② $B_j^+ = B_j \cap \{x_2 > 0\}$, $B_j^- = B_j \cap \{x_2 < \varepsilon_j\}$,

where ε_j is chosen s.t. $|B_j^-| + |B_j^+| \leq |B_j| + \frac{\varepsilon}{2^{j+1}}$

Another Approach (Tao) : Tao Ex. 7.4.3

For A = open box in $\mathbb{R}^n$, prove

$$m^*(A) = m^*(A_+) + m^*(A_-)$$

* $m^*(A) = |A| \Rightarrow m^*(A_+) = |A_+| \cdots$ by direct calculation

Then for general A, for any $\varepsilon > 0$, find $\{B_j\}$ cover of A s.t.

$$m^*(A) + \varepsilon \geq \sum |B_j|$$

may not be open

Then define $B_j^+ = B_j \cap \{x_n > 0\}$, $B_j^- = B_j \cap \{x_n \leq 0\}$

$\Rightarrow \sum |B_j| = (\sum |B_j^+|) + (\sum |B_j^-|)$

$\Rightarrow$ Since $A_+ \subset \bigcup B_j^+$, $m^*(A_+) \leq \sum m^*(B_j^+) = \sum |B_j^+|$

$A_- \subset \bigcup B_j^-$, $m^*(A_-) \leq \sum m^*(B_j^-) = \sum |B_j^-|$

$\Rightarrow m^*(A_+) + m^*(A_-) \leq m^*(A) + \varepsilon$

Lemma 7.4.4 (Properties of measurable sets).

(a) *If E is measurable, then $\mathbf{R}^n \backslash E$ is also measurable.* $= E^c$

(b) *(Translation invariance) If E is measurable, and $x \in \mathbf{R}^n$, then $x + E$ is also measurable, and $m(x + E) = m(E)$.*

(c) *If E_1 and E_2 are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.* $\{E_1^c \cap E_2^c\}^c$

(d) *(Boolean algebra property) If $E_1, E_2, \ldots, E_N$ are measurable, then $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are measurable.*

(e) *Every open box, and every closed box, is measurable.* (or half)

(f) *Any set E of outer measure zero (i.e., $m^*(E) = 0$) is measurable.*

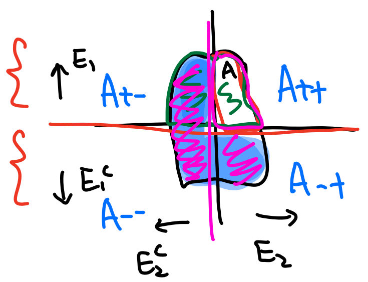
(a) True by definition: same test for E and E^c

(b) Outer measure is translation invariant

$\Rightarrow \forall A \subset \mathbb{R}^n,\ m^*(A) \overset{?}{=} m^*(A \cap (x+E)) + m^*(A \cap (x+E)^c)$

$\Leftrightarrow m^*(A - x) \overset{?}{=} m^*((A-x) \cap E) + m^*((A-x) \cap E^c)$

(c) W.T.S.: $\forall A \subset \mathbb{R}^n,\ m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \backslash (E_1 \cap E_2))$ $(*)$



Let $A_{++} = A \cap E_1 \cap E_2$

$A_{+-} = A \cap E_1 \cap E_2^c$

$A_{-+} = A \cap E_1^c \cap E_2$

$A_{--} = A \cap E_1^c \cap E_2^c$

$\Rightarrow A = A_{++} \sqcup A_{+-} \sqcup A_{-+} \sqcup A_{--}$

$(*) \Leftrightarrow \boxed{m^*(A) = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})}$

We can show:

$\rightarrow m^*(A) = m^*(A_{+-} \cup A_{++}) + m^*(A_{-+} \cup A_{--})$ using E_1 measurable

$\rightarrow m^*(A_{+-} \cup A_{++}) = m^*(A_{+-}) + m^*(A_{++})$ } using E_2 measurable

$\rightarrow m^*(A_{-+} \cup A_{--}) = m^*(A_{--}) + m^*(A_{-+})$

Still need to show:

$$m^*(A_{+-} \cup A_{-+} \cup A_{--}) = m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$$

(d) Induction on #(operand)

(e) E.g. $[a,b] = [a,\infty) \cap (-\infty, b]$

$$[a,\infty) = a + [0,\infty), \quad (-\infty,b] = b + (-\infty, 0]$$

Boxes are intersections of (translated) half spaces

(f) Show: $\forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$

need: $m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$

$$m^*(A \cap E) \leq m^*(E) = 0 \Rightarrow m^*(A \cap E) = 0$$

$\Leftrightarrow m^*(A) \geq m^*(A \setminus E)$ by monotonicity