- Remark about HW1

want: $m^*(\text{any box}) = \text{vol}(\text{box})$ (About Lebesgue Measure)

We know: $m^*(\text{closed box}) = \text{volume}$

$m^*(\text{open box}) = \text{volume}$

If we have any "half open half closed" box, B,

$B^\circ \subset B \subset \bar{B}$, then

$\text{vol}(B^\circ) = m^*(B^\circ) \le m^*(B) \le m^*(\bar{B}) = \text{vol}(\bar{B})$

$\Rightarrow m^*(B) = \text{vol}(\bar{B}) = \text{vol}(B^\circ) = \prod (b_i - a_i)$

Corollary 7.4.7. *If $A \subseteq B$ are two measurable sets, then $B \backslash A$ is also measurable, and*

$$m(B \backslash A) = m(B) - m(A).$$

$\boxed{\text{Pf}}$ • $B \backslash A = B \cap A^c$

A is measurable $\Rightarrow$ A^c measurable

B, A^c measurable $\Rightarrow$ $B \cap A^c$ measurable

• W.T.S.: $m^*(B) = m^*(A) + m^*(B \backslash A)$

This follows from measurability of A, applied to test set B

$\Rightarrow m^*(B) = m^*(\underbrace{B \cap A}_{=A}) + m^*(B \cap A^c)$

Lemma 7.4.8 (Countable additivity). *If $(E_j)_{j \in J}$ are a countable collection of* disjoint *measurable sets, then $\bigcup_{j \in J} E_j$ is measurable, and $m(\bigcup_{j \in J} E_j) = \sum_{j \in J} m(E_j)$.*

$\boxed{\text{Pf}}$ • W.T.S. $\forall A \subset \mathbb{R}^n$,

(*) $m^*(A) = m^*(A \cap E) + m^*(A \backslash E)$

• Define $F_N = \bigsqcup_{j=1}^{N} E_j$. We know:

1) F_N is measurable

2) $m^*(F_N) = \sum_{j=1}^{N} m^*(E_j)$

- If we replace E by F_N, since $E \supset F_N$, $E^c \subset F_N^c$. Then

$$m^*(A \cap E) \geq m^*(A \cap F_N)$$
$$m^*(A \cap E^c) \leq m^*(A \cap F_N^c)$$

- To prove (*), we need "$\leq$" and "$\geq$"

"$\leq$" : finite sub-additivity

"$\geq$" :

$$\begin{aligned} m^*(A \cap E) &\leq \sum_{j=1}^{\infty} m^*(A \cap E_j) \quad \text{by countable sub-additivity} \\ &= \sup_N \sum_{j=1}^{N} m^*(A \cap E_j) \\ &= \sup_N m^*(A \cap F_N) \quad \text{by finite additivity} \end{aligned}$$

Therefore,

$$\begin{aligned} m^*(A \cap E) + m^*(A \cap E^c) &\leq [\sup_N m^*(A \cap F_N)] + m^*(A \cap E^c) \\ &\leq \sup_N (m^*(A \cap F_N) + m^*(A \cap E^c)) \\ &\leq \sup_N (m^*(A \cap F_N) + m^*(A \cap F_N^c)) \\ &= \sup_N [m^*(A)] \\ &= m^*(A) \end{aligned}$$

☆ change infinite union to discrete union then additivity + limit

- $m^*(E) \leq \sum_j m^*(E_j)$ by sub-additivity

$$m^*(E) \geq m^*(F_N) = \sum_{j=1}^{N} m^*(E_j)$$

sup over N, we have

$$m^*(E) \geq \sum_{j=1}^{\infty} m^*(E_j)$$

$$\Rightarrow m^*(E) = \sum_{j=1}^{\infty} m^*(E_j)$$

Lemma 7.4.9 (σ-algebra property)**.** *If $(\Omega_j)_{j\in J}$ are any countable collection of measurable sets (so J is countable), then the union $\bigcup_{j\in J}\Omega_j$ and the intersection $\bigcap_{j\in J}\Omega_j$ are also measurable.*

$\boxed{\text{Pf}}$ • Consider $\Omega = \bigcup_{j=1}^{\infty}\Omega_j$. Let

Disjointize union

$$\Omega_N = \bigcup_{j=1}^{N}\Omega_j,\quad E_N = \Omega_N \setminus \Omega_{N-1}$$

Then, $\{\Omega_N\}$ are measurable, $\{E_N\}$ are measurable. Since

$$\Omega = \bigcup_{j=1}^{\infty} E_j,$$

Ω measurable.

• $\bigcap_{j=1}^{\infty}\Omega_j = \left(\bigcup_{j=1}^{\infty}\Omega_j^c\right)^c$. Since complement & countable union preserve measurability, $\bigcap_{j=1}^{\infty}\Omega_j$ is measurable.

Lemma 7.4.10. *Every open set can be written as a countable or finite union of open boxes.*

Recall : Topology

- Topology for a metric space (X, d)
  - open ball $B(x,r) = \{y\in X \mid d(y,x) < r\}$, $x\in X$, $r\in\mathbb{R}^+$
  - open sets in X are generated from open balls by taking finite intersection & arbitrary union.
  - equivalently, $U\subset X$ is open iff $\forall x\in U$ $\exists r>0$ s.t. $B(x,r)\subset U$
- Topology for product space

  If X, Y are topological spaces, then $X\times Y$ can be endowed w/

  ? product topology, i.e. $W\subset X\times Y$ is open, if $\forall (x,y)\in W$, $\exists U\subset X$, $V\subset Y$ open s.t. $(x,y)\in U\times V\subset W$

- Topology on $\mathbb{R}^2$:
  - Can be generated by open balls (using Euclidean metric on $\mathbb{R}^2$)
  - Can be generated by open box

$\boxed{\text{Pf}}$ • Consider "rational boxes". A box $\prod(a_i, b_i)$ is rational if

$a_1, b_1, \dots, a_n, b_n \in \mathbb{Q}$.

? • {The collection of rational boxes} $\subset \mathbb{Q}^{2n}$ is countable

↳ Since $\mathbb{Q}$ is countable, finite product of countable set is countable, and subset of countable set is countable.

• Suffice to show that, every open set in $\mathbb{R}^n$ is a union of rational boxes

Want to find a rational open box B s.t. $x \in B \subset U$

Since U is open, $\exists r > 0$ s.t. $x \in B(x, r) \subset U$

Claim: $\exists$ rational box B s.t. $x \in B \subset B(x, r)$

Lemma 7.4.11 (Borel property). *Every open set, and every closed set, is Lebesgue measurable.*

$\boxed{\text{Pf}}$ • open boxes are measurable

• an open set is a countable union of open boxes

Alternative Definition of Measurable Set:

A subset $E \subset \mathbb{R}^n$ is measurable, if $\forall \varepsilon > 0$, $\exists$ exists an open set U s.t. $U \supset E$,

$m^*(U \setminus E) < \varepsilon$ $\qquad$ $m^*(U_1 \setminus E_1) < \frac{\varepsilon}{2}$

Original Def.

Definition 7.4.1 (Lebesgue measurability). Let E be a subset of $\mathbf{R}^n$. We say that E is *Lebesgue measurable*, or *measurable* for short, iff we have the identity

$$m^*(A) = \underline{m^*(A \cap E) + m^*(A \backslash E)}$$

for every subset A of $\mathbf{R}^n$. If E is measurable, we define the *Lebesgue*

$$m^*(A) = m^*(E) + m^*(A \backslash E)$$

$$m^*(A) \geqslant m^*(E) \Rightarrow m^*(A) - m^*(E) \geqslant 0$$

$$m^*(A \backslash E) \geqslant 0$$

Covering $\{B_j\}$ s.t.

$$\{B_j^A\} \supset \{B_j^E\}$$

$$m^*(E) \leqslant \sum_j |B_j^E| \leq \sum_j |B_j^A| \leqslant m^*(A) + \varepsilon$$