Abstract Measure Theory

S set. 2^S = the set of all subsets in S

$\hookrightarrow \mathrm{Map}(S, \{0,1\})$

- (σ-Algebra) A subset $M_S \subset 2^S$ is called a <u>σ-alg.</u> if:
  1) $\phi \in M_S$
  2) If $A_1, A_2, \dots \in M_S$, countable collection, then $\bigcup_{n=1}^{\infty} A_n \in M_n$
  3) $A \in M_S \Rightarrow A^c \in M_S$

- (Measure) A <u>measure</u> ω on (S, M_S) is a fxn.

  $\omega : M_S \longrightarrow [0, +\infty]$ s.t.
  1) $\omega(\phi) = 0$
  2) (countable addition) If $A_1, A_2, \dots \in M_S$ disjoint, then

     $$\omega(\cup A_n) = \sum \omega(A_n)$$

  $\rightarrow$ (S, M_S, ω) is called a measure space

- (Measurable Fxn.) Given (X, M_X) and (Y, M_Y), we say a measurable map/fxn. from X to Y, is

  $f : X \rightarrow Y$ s.t. $\forall E \in M_Y$, we have $f^{-1}(E) \in M_X$

  $\rightarrow$ If
  $$\begin{tikzcd} (X, M_X) \arrow[r, "f"] \arrow[rr, bend right, "g\circ f \text{ is measurable}"'] & (Y, M_Y) \arrow[r, "g"] & (Z, M_Z) \end{tikzcd}$$

- If S is a topological space, i.e. $\mathcal{T}_S \subset 2^S$ collection of open subsets

  ?

  - $\phi \in \mathcal{T}_S$
  - If $U_1, \dots, U_n \in \mathcal{T}_S \Rightarrow U_1 \cap \dots \cap U_n \in \mathcal{T}_S$
  - Arbitrary union of open is open

then there is a minimal σ-alg., containing $\mathcal{T}_S$: $\langle \mathcal{T}_S \rangle$

↳ Borel σ-alg.

⇓ ✗

Lesbegue

- If S is a topological space,
  - If $U_1, U_2, \ldots$ is a countable collection on open sets, then $\bigcap_{i=1}^{\infty} U_n$ is called a G_δ set.
  - If $F_1, F_2, \ldots$ are closed sets, then $\bigcup_{n=1}^{\infty} F_n$ is called a F_σ set

- Let S be a set, an outer measure ω on S is a fxn.

  $\omega : 2^S \to [0, \infty]$ s.t.

  ① $\omega(\phi) = 0$

  ② If $A \subset B$, then $\omega(A) \le \omega(B)$

  ③ (Countable Sub-Additivity) If $A_1, A_2, \ldots$, $\omega(\bigcup_n A_n) \le \sum_{n=1}^{\infty} \omega(A_n)$

→ Construction: Given ω, $M_S \subset 2^S$, $E \in M_S$ if $\forall X \subset S$,

$\omega(X) = \omega(X \cap E) + \omega(X \cap E^c)$

$\boxed{\text{Thm}}$ (5) ?

- M_S is a σ-alg.
- ω on M_S satisfies countable additivity. (S, M_S, ω) is a measure space.

- A subset $E \subset S$, w/ $\overset{\text{outer measure}}{\omega}(E) = 0$, is called a "zero set" or "null set"

$\boxed{\text{Lemma}}$ If E is a null set

① If $E' \subset E$, then E' is null

② $\forall A \subset S$, $\omega(A \cup S) = \omega(A)$

③ $\forall A \subset S$, $\omega(A \cap E^c) = \omega(A)$

④ $\omega(E)=0 \Rightarrow E$ measurable

⑤ If Z is a null set, then F measurable $\Leftrightarrow F\cup Z$ measurable

$\boxed{\text{Pf}}$ ① By monotonicity of outer-measure

② $\omega(A\cup E) \le \omega(A) + \omega((A\cup E)\cap A^c)$

$\geq$

$\omega(A)$

$= \omega(A) + \omega(E\cap A^c)$

$= \omega(A)$

$\Rightarrow \omega(A) = \omega(A\cup E)$

④ $\forall A\subset S$, W.T.S. $\omega(A) = \omega(A\cap E) + \omega(A\cap E^c)$

$= 0 + \omega(A)$

⑤ $\Rightarrow$ W.T.S: $\forall A\subset S$,

$$\omega(A) = \omega(A\cap(F\cup Z)) + \omega(\overset{A\cap(F^c\cap Z^c)}{\overset{\shortparallel}{A\cap(F\cup Z)^c}})$$

$$= \omega((A\cap F)\cup(A\cap Z)) + \omega(A\cap F^c\cap Z^c)$$

$$= \omega(A\cap F) + \omega(A\cap F^c)$$

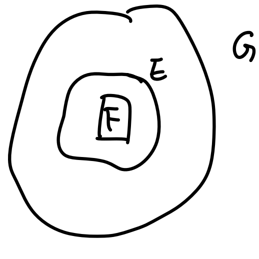
$= \omega(A)$ since F measurable

- $\Rightarrow$ Adding / subtracting null set does not affect measurability and measure
- E.g. $\{0\}\times\mathbb{R}\subset\mathbb{R}^2$ has measure 0.

$\boxed{\text{Pf}}$ $\forall \varepsilon>0$, $\exists$ open box cover of $\{0\}\times\mathbb{R}$ s.t. $\sum|B_j|<\varepsilon$

$B_j = \left(-\frac{\varepsilon}{2^{2n+2}}, \frac{\varepsilon}{2^{2n+2}}\right)\times(-2^n, 2^n)$

$|B_n| = \frac{\varepsilon}{2^n} \Rightarrow \sum|B_n| = \varepsilon$



$\boxed{\text{Thm.}}$ ($\mathbb{R}^n$, Lesbegue measure)

$\forall E\subset\mathbb{R}^n$, E is Lebegue measure

$\Leftrightarrow \exists\ F_\sigma$ set F, and G_δ set G, s.t. $F\subset E\subset G$, and $m(G\setminus F)=0$

↳ countable union of closed (∩ open)

$\boxed{\text{Pf}}$ Assume E is bounded

$\Rightarrow$ Assume E is Lebegue, and $E \subset \overset{(-a,a)^n}{R}$ big box

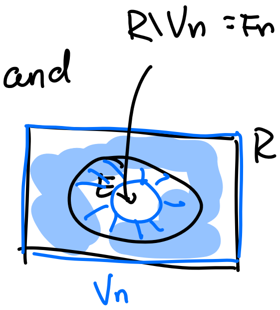
$\forall n$, $\exists$ box covering $\{B_j^n\}$ of E, s.t.

$\sum |B_j^{(n)}| < m(E) + \frac{1}{n}$, let $U_n = \bigcup_j B_j^{(n)}$

Similarly, let $V_n \supset R \setminus E$, s.t. V_n open, $V_n \subset R$ and

? $\quad m(R \setminus E) \le m(V_n) \le m(R \setminus E) + \frac{1}{n} \quad (2)$

$m(E) \le \underline{m(U_n)} \le m(E) + \frac{1}{n} \quad (1)$



Define

$G = \bigcap_{n=1}^{\infty} U_n$, $\quad F_n = R \setminus \overset{\text{closed}}{V_n}$, $F_n \subset E$, $F = \bigcup F_n$

$(2) \Leftrightarrow m(F_n) \le \underline{m(E)} \le m(F_n) + \frac{1}{n} \quad (2')$

$(1) + (2') \Rightarrow m(G \setminus F) \le m(U_n \setminus F_n) \le \frac{2}{n} \quad \forall n$

$\Leftarrow$ "sandwiched" b/n 2 sets