Recall :

- Lebesgue criterion for measurability:

  $E \subset \mathbb{R}^n$ is measurable if $\exists$ a G_δ-set G, and an F_σ-set F s.t. $G \supset E \supset F$, and $m(G \setminus F) = 0$.

  - G is an "null" on E $\Big\}$ unique up to a null set
    F a "kernel" on E

Lemma

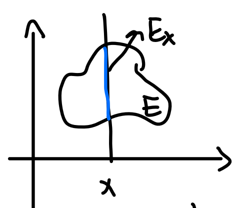
- If $G_1, G_2, \dots$ are G_δ set. Then $\cap G_i$ is a G_δ set

  $\boxed{\text{Pf}}$ $G_1 = \bigcap_{j=1}^{\infty} \underset{\swarrow\text{open}}{U_{1j}}, \dots$

  $G = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} U_{i,j}$ still intersection on countably many open

- $F_1, F_2, \dots F_\sigma$, then $\bigcup_{i=1}^{\infty} F_i$ is F_σ

- Product & Slices



$\boxed{\text{Thm}}$ If $E \subset \mathbb{R}^n$, $F \subset \mathbb{R}^k$ are measurable sets, then $m(E \times F) = m(E) \cdot m(F)$. If $m(E) = m(F) = 0$, then $m(E \times F) = 0$.

hyperplane

$\boxed{\text{Pf}}$ $m(\{a\} \times \mathbb{R}) = 0$

$(n=1, k=1)$

Lemma : If $E \subset \mathbb{R}$ has $m(E) = 0$, then $m(E \times \mathbb{R}) = 0$.

$\boxed{\text{Pf}}$ $\forall\, n > 0$ integer, we will cover E, by collection on boxes, w/ total area $\frac{\varepsilon}{2^{2n}}$, $\{B_{n,i}\}_{i=1}^{\infty}$,

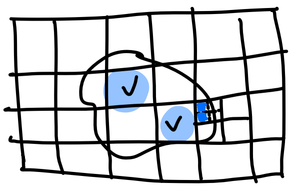
$\bigcup_{i=1}^{\infty} B_{n,i} \supset E$, $\sum_{i=1}^{\infty} |B_{n,i}| \le \frac{\varepsilon}{2^{2n+1}}$

Define $\tilde{B}_{n,i} = B_{n,i} \times (-2^n, 2^n)$, $\sum_{i=1}^{\infty} |\tilde{B}_{n,i}| \le \frac{\varepsilon}{2^n}$,

$$\sum_{n=1,i=1}^{\infty}\sum |\tilde{B}_{n,i}| \le \varepsilon$$

<u>Lemma</u>: Any open set $U \subseteq \mathbb{R}^n$ can be written as a countable disjoint union of open boxes, and a measure zero set.

$\boxed{\text{pf}}$ $(n=2)$



- Consider unit open squares in $\mathbb{R}^2$.

  $(a, a+1) \times (b, b+1)$, $a, b \in$?

- For size 1 open boxes contained in U, we take them.

  For size ___ . that $\cap U = \phi$, ignore them.

  For any open box B s.t. $B \cap U \neq \phi$, we further divide B into 4 size $\frac{1}{2}$ piece.

<u>Lemma</u>: $m(E \times F) = m(E) \times m(F)$, for

- E, F are open box

$\Downarrow$

- E, F are boxes
- E, F are open set

$\boxed{\text{pf}}$ $E = \bigsqcup_{i=1}^{\infty} B_i + Z$

$F = \bigsqcup_{i=1}^{\infty} B_i' + Z'$, B_i, B_i' open boxes

Hence,

$$m(E \times F) = m\left(\left(Z + \bigsqcup_{i=1}^{\infty} B_i\right) \times \left(Z' + \bigsqcup B_i'\right)\right)$$

$$= m\left(\bigsqcup_{i=1}^{\infty} B_i \times \bigsqcup_{i=1}^{\infty} B_i'\right)$$

?

$$= \sum_{i,j} m(B_i) \times m(B_j') = \left(\sum_i m(B_i)\right) \times \left(\sum_j (m(B_j')\right)$$

For E, F measurable subsets on $\mathbb{R}$, now prove $m(E\times F) = m(E)\times m(F)$

$\boxed{\text{pf}}$ · Assume E, F are bounded. $E\subset R_1$, $F\subset R_2$, R_i open boxes

· Let $H_E \supset E \supset K_E$, $H_F \supset F \supset K_F$

null kernel

$\Rightarrow m(H_E\setminus K_E) = 0$, $m(H_F\setminus K_F) = 0$

· Since H_E & H_F are G_δ set, $m(H_E\times H_F) = m(H_E)\times m(H_F)$

$\boxed{\text{pf}}$?

· <u>Claim</u> : 1) $H_E\times H_F$ is G_δ, $K_E\times K_F$ is F_σ

2) $m(H_E\times H_F\setminus K_E\times K_F) = 0$

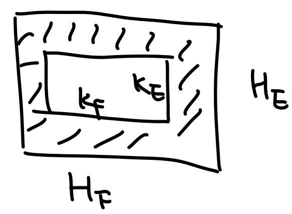
$\boxed{\text{pf}}$ 1) · $H_E = \bigcap U_n$, $H_F = \bigcap V_n$

$$H_E\times H_F = \bigcap_i\bigcap_j U_i\times V_j \Rightarrow G_\delta$$

2) · $H_E\times H_F\setminus K_E\times K_F$

$\subset (H_E\setminus K_E)\times H_F \cup H_E\times(H_F\setminus K_F)$

↳ measure 0.



$\Rightarrow E\times F$ is measurable

$$m(E\times F) = m(H_E\times H_F) = m(H_E)\times m(H_F) = m(E)\times m(F)$$

Let $E\subset\mathbb{R}^n\times\mathbb{R}^b$, for any $x\in\mathbb{R}^n$, let $E_x = E\cap\{x\}\times\mathbb{R}^k \subset \{x\}\times\mathbb{R}^k \simeq \mathbb{R}^k$

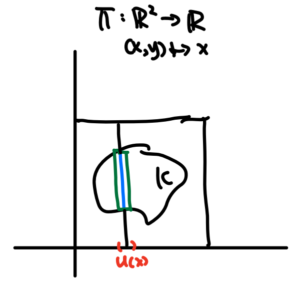
$\boxed{\text{Thm.}}$ If $Z = \{x\in\mathbb{R}^n \mid m_{\mathbb{R}^k}(E_x)\neq 0\}$ is measure zero in $\mathbb{R}^n$, then $m(E) = 0$.

$\boxed{\text{pf}}$ Let $\tilde{E} = E\setminus Z\times\mathbb{R}^k$, then $m(Z\times\mathbb{R}^k) = 0$

$\Rightarrow m(\tilde{E}) = m(E)$. Suffice to show $m(\tilde{E}) = 0$.

Replace E by $\tilde{E}$, and assume $Z = \phi$, i.e. $m(E_x) = 0\ \forall x$

?

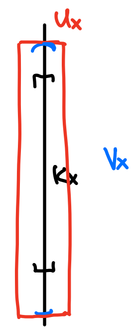


- Assume E is a bounded subset in $\mathbb{R}^2$. $(n=1, k=1)$
- Assume $E \subset [0,1]^2$, $m(E_x)=0$ $\forall x$
- $\forall \varepsilon>0$, W.T.S. : $m(E)<\varepsilon$

① Find $K \subset E$ closed s.t. $m(E \setminus K) \le \frac{\varepsilon}{2}$.

Then K is closed & b.d. $\Rightarrow$ K compact & $m(K_x)=0$ $\forall x$.

② Cover K by boxes of total area $\le \frac{\varepsilon}{2}$ $\Rightarrow m(K) \le \frac{\varepsilon}{2}$



- $\forall x \in \mathbb{R}$, if $K_x = \emptyset$, we can find $V(x) \subset \mathbb{R}$ open s.t.

  $m(V(x)) < \frac{\varepsilon}{2}$, $V(x) \supset K_x$

- Claim: $\exists\, U(x) \ni x$ s.t. $U(x) \times V(x) \supset \pi^{-1}(U(x)) \cap K$

  i.e. $\forall \tilde{x} \in U(x)$, want $V_x \supset K_{\tilde{x}}$

$\boxed{\text{Pf}}$ Suppose $\nexists\, U(x) \ni x$ w/ this property

i.e. $\forall \varepsilon>0$, $\exists\, y$ w/ $|y-x|<\varepsilon$ s.t. $V_x \not\supset K_{\tilde{x}}$ $\exists\, (\tilde{x},\tilde{y}) \subset K_{\tilde{x}}$

s.t. $\tilde{y} \notin V_x$

Then $\exists\, (\tilde{x}_n, \tilde{y}_n) \in K$ s.t. $\tilde{x}_n \to x$, and $\tilde{y}_n \notin V(x)$ by

passing to subsequence, we may assume

$$(\tilde{x}_n, \tilde{y}_n) \to (x,y) \in K_x$$

So $\tilde{y}_n \to y$, but $\tilde{y}_n \in V_x^c \Rightarrow y \in V_x^c$

Contradiction: $y \in K_x \subset V_x$

$\Rightarrow$ Thus, $\forall x \in \mathbb{R}$, $\exists\, V_x \supset K_x$, $m(V_x) < \frac{\varepsilon}{2}$

$\exists\, U_x \ni x$, s.t. $U_x \times V_x \supset \pi^{-1}(U_x) \cap K$

- $K \subset \bigcup_{x \in \mathbb{R}} V_x \times U_x$, K compact

  $\Rightarrow$ can pass to finite subcover

  $K \subset V_{x_1} \times U_{x_1} \cup \cdots \cup V_{x_N} \times U_{x_N}$

Define $U_i = U_{x_i} \setminus (U_{x_1} \cup U_{x_2} \cup U_{x_3} \cdots)$

$\Rightarrow K \subset V_{x_1} \times U_{x_1} \sqcup V_{x_2} \times U_2 \sqcup \cdots$

$\Rightarrow \sum m(V_{x_i} \times U_{x_i}) = \sum m(V_{x_i}) \times m(U_i)$

$\leq \frac{\varepsilon}{2} \sum m(U_i) \leq \frac{\varepsilon}{2} \times 1 < \frac{\varepsilon}{2}$