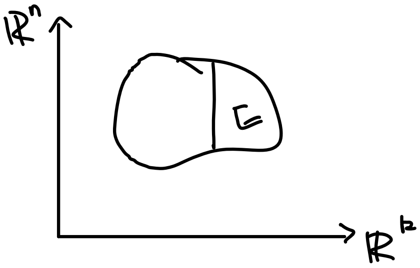
- Recall: <u>Slice Thm.</u>




$E \subset \mathbb{R}^k \times \mathbb{R}^n$

$x \in \mathbb{R}^k, \quad E_x = E \cap \{x\} \times \mathbb{R}^n \subset \mathbb{R}^n$

$\boxed{\text{Thm.}}$ $m(E) = 0 \iff$ almost every slice $E_x$ has measure zero

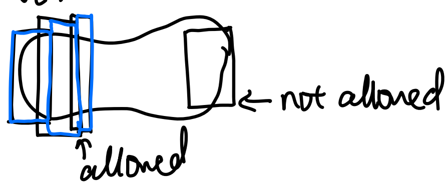
I.e. Let $Z := \{x \mid m(E_x) > 0\}$, then $m(Z) = 0$, $Z \subset \mathbb{R}^k$

$\Leftarrow$ ① May assume $Z = \phi$, $E$ is bounded

② Use inner approximation a $E$ by compact subset $K$

③ Cover $K$ by many open boxes $U_i \times V_i$,

then disjointized



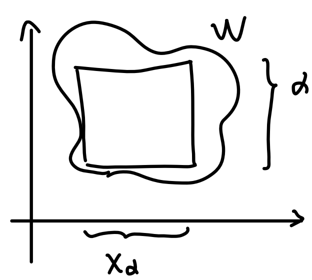

④ $m(K) \leq \sum_i m(U_i)\, m(V_i) < \varepsilon$

$\sum m(U_i) < 1 < \varepsilon$

⑤ $m(E) < m(K) + \varepsilon < 2\varepsilon$

$\Rightarrow$ <u>Lemma</u>: $\forall W$ open, $\forall \alpha > 0$, let $X_\alpha = \{x \in \mathbb{R}^k \mid m(W_x) > \alpha\}$, $W \subset \mathbb{R}^k \times \mathbb{R}^n$, b.d.

then $m_{n+k}(W) \geq m_k(X_\alpha) \cdot \alpha$



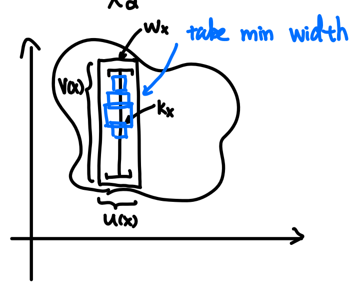

$m(W) \geq m(\square)$
$= m(X_\alpha) \cdot \alpha$

$\boxed{\text{pf}}$ ① $\forall x \in X_\alpha$, get a compact set

can get arbitrarily close

$K_x \subset W_x$, $m(K_x) > \alpha$

$\{x\} \times K_x \subset \underset{\mathbb{R}^k}{U(x)} \times \underset{\mathbb{R}^n}{V(x)} \subset W$




$\Rightarrow X_\alpha$ is open since $U(x) \subset X_\alpha$, indeed,

$\forall x' \in U(x)$, $W_{x'} \supset V(x) \supset K_x$

$\Rightarrow m(W) \geq m(W_{x'}) \geq m(K_x) > \alpha$

② $\forall K' \subset\subset X_\alpha$, since $X_\alpha \subset \bigcup_{x \in X_\alpha} U(x)$,

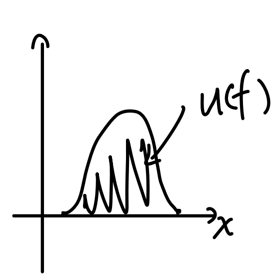
we can have finite subcover for $K'$

$K' \subset U(x_1) \cup U(x_2) \cdots \cup U(x_n)$

$u_1 = U(x_1)$, $u_2 = U(x_2) \setminus U(x_1)$, ... $u_i$ disjoint

$m(W) \geq m(\bigsqcup_{i=1}^{N} u_i \times V(x_i)) \geq \sum m(u_i) \times \alpha \geq m(K') \cdot \alpha$

## Lebesgue Integral (Pugh 6.6)




Let $f : \mathbb{R} \to [0, \infty)$

Def : Undergraph $U(f) := \{(x,y) \mid 0 \leq y < f(x)\}$

$f$ is measurable if $U(f)$ is a measurable subset

$\Rightarrow$ $\int f := m(U(f))$ (possibly $= +\infty$)

* If $\int f < \infty$, we say $f$ is integrable

• Notation: • a.e. : almost everywhere = "up to a measure zero set"

E.g. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases} \Rightarrow f(x) = 0$ a.e.

• $A_n \nearrow A$ : $A_n \subset A_{n+1} \subset \cdots$, $A = \bigcup A_n$

**Thm. 27.** : Let $f_n : \mathbb{R} \to [0,\infty)$ be a seq. of measurable fxn., and $f_n \nearrow f$ a.e. as $n \to \infty$. Then $\int f_n \nearrow \int f$

(There is a null set $Z \subset \mathbb{R}$ s.t. $\forall x \in \mathbb{R} \setminus Z$ $\lim_{n\to\infty} f_n(x) = f(x)$, $f_{n+1}(x) \geq f_n(x)$.) ?

Pf $f_n \to f \Rightarrow U(f_n) \nearrow U(f) \xRightarrow[\text{Thm.5}]{} m(U(f_n)) \nearrow m(U(f))$

• Completed graph : $\hat{U}(f) := \{(x,y),\ 0 \leq y \leq f(x)\}$

↖ fiberwise closed interval

* $U(f)$ is measurable $\Leftrightarrow$ $\hat{U}(f)$ is measurable

$m(u(f)) = m(\hat{u}(f))$

**Fact** (Pugh 6.3)

If $T: \mathbb{R} \to \mathbb{R}^n$ affine linear transformation

$T(x) = Ax + b$

$E \subset \mathbb{R}^n$ is measurable, then

$m(T(E)) = |T|\, m(E)$, where $|T| = |\det(A)|$

$\boxed{\text{Pf}}$ $\Rightarrow$ $\forall n > 0$, integer

$$u(f) \subset \hat{u}(f) \subset \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1+\frac{1}{n} \end{pmatrix} u(f) + \mathbb{R} \times \{0\}}_{R_n(f)}$$

stretch

$\int f_n = \infty$

$\int f = 0$

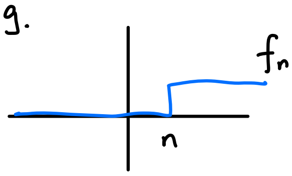
$f_n(x) = \begin{cases} 0, & x < n \\ 1, & x > n \end{cases}$

$\bigcap_{n=1}^{\infty} R_n(f)$ is measurable

$m(\cap R_n(f)) = \lim_{n\to\infty} m(R_n(f))$

$= \lim (1 + \frac{1}{n})\, m(u(f)) = m(u(f))$

E.g.




$\boxed{\text{Thm}}$ If $f_n : \mathbb{R} \to [0, \infty)$ is a seq. of <u>integrable</u> fxn., that

$f_n \searrow f$ a.e., then $\int f_n \searrow \int f$

$\boxed{\text{Pf}}$ $m(U(f)) = m(\hat{u}(f)) \swarrow m(\hat{u}(f_n)) = m(u(f_n)) = \int f_n < \infty$

$\int f$ $\parallel$

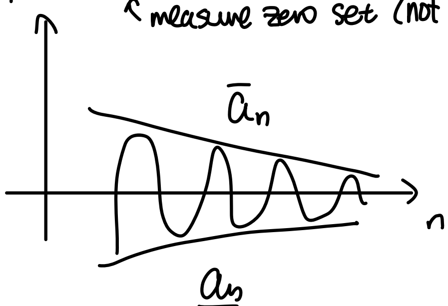
$u(f_n) \overset{\times}{\searrow} u(f)$

$f_n$: $\frac{1}{n}$ $\quad f = 0$, $u(f) = \emptyset$, $u(f_n) =$ ← measure zero set (not empty)

**Recall:** If $a_n$ is a bounded sequence,

$\bar{a}_n := \sup\{a_m : m \geq n\}$

$\underline{a_n} := \inf\{a_m : m \geq n\}$




Now, if $f_n(x)$ is a seq. of fxns,

$\bar{f}_n(x) = \sup\{f_m(x), m \geq n\}$, $\underline{f_n}(x) = \cdots$

$\forall x$, $\lim \underline{f_n}(x) = \lim\inf f_n(x) = \lim f_n(x) = \lim\sup f_n(x) = \lim \bar{f}_n(x)$

**Prop** : (1) $U(\bar{f}_n) = \bigcup_{k \ge n} U(f_k)$ $\implies$ $\underline{f}_n(x) \le f_n(x) \le \bar{f}_n(x)$, all measurable

(2) $\hat{u}(\underline{f}_n) = \bigcap_{k \ge n} \hat{u}(f_k)$

$\boxed{\text{pf}}$ (1) $\forall x$, $[0, \sup_{k \ge n} f_k(x)) = \bigcup_{k \ge n} [0, f_m(x))$

(2) $\forall x$, $[0, \inf_{k \ge n} f_k(x)] = \bigcap_{k \ge n} [0, f_k(x)]$

$\boxed{\text{Thm.}}$ (Dominant Convergence)

$f_n, f : \mathbb{R} \to [0, \infty)$

- Suppose we have a seq. of $f_n$ measurable, $f_n \to f$ a.e.
- $\exists\, g : \mathbb{R} \to [0, \infty)$, s.t. $g(x) \ge f_n(x)$ a.e., $\int g < \infty$

Then $\int f_n \to \int f$

$\boxed{\text{pf}}$ Since $u(f_n) \subset U(g)$, $m(U(f_n)) \le m(U(g)) < \infty$

$u(\underline{f}_n) \subset u(f_n) \subset \hat{u}(\bar{f}_n)$

$\underline{f}_n \nearrow f$, $\bar{f}_n \searrow f$ $\underset{+\int \bar{f}_n < \infty}{\Longrightarrow}$ $\lim \int \underline{f}_n = \int f$, $\lim \int \bar{f}_n = \int f$

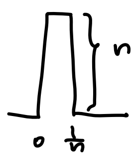
$\lim \int f_n$

$\bar{f}_n \le g \Rightarrow \bar{f}_n$ integrable

- E.g. $f_n(x) = \mathbb{1}_{[n, n+1]}(x)$, $f_n(x) \to 0 = f$ pt.wise ($\nexists\, g$)

$\int f_n = 1$, $\int f = 0$

- E.g. $f_n = n \cdot \mathbb{1}_{[0, \frac{1}{n}]}(x)$

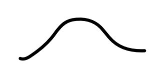


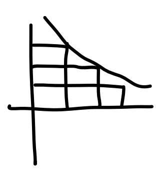

or $f_n = \frac{1}{n} \mathbb{1}_{[0, n]}$

$f_n \to 0$

$\lim f_n(x) = \begin{cases} 0 & x \ne 0 \\ \infty & x = 0 \end{cases}$

( $\nexists\, g$, perhaps





$g = \frac{1}{x}$ (not integrable )

$\Rightarrow f_n \to 0 = f$ a.e.

$\int f_n = 1$, $\int f = 0$