Q: If $f_n \to f$ a.e. and $\int f_n < M$ ... ?

Claim: $\int f < M$?

no

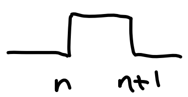
pf:

- $\underline{f}_n$, $\overline{f}_n$: measurable

  $\underline{f}_n$: integrable ($\underline{f}_n \leq f_n$)

- E.g. the running hump

  f_n:



$\underline{f}_n(x) = 0$, $\overline{f}_n(x) = \infty$ $\Rightarrow$ $\int \underline{f}_n = \int f$

$f = 0$ $\qquad\qquad\qquad\qquad$ $\int \overline{f}_n \neq \int f$

Fatou's Lemma:

If $f_n : \mathbb{R} \to [0,\infty)$ measurable, then $\int \liminf f_n \leq \liminf \int f_n$.

pf: Def.: $\liminf f_n = \lim \underline{f}_n = f$

↳ We know by monotone convergence, $\int \lim \underline{f}_n = \lim \int \underline{f}_n$

$\Rightarrow \underline{f}_n \leq f_m \ \forall m \geq n$, $\int \underline{f}_n \leq \int f_m \ \forall m \geq n$

$\Rightarrow \int \underline{f}_n \leq \inf_{m \geq n} \int f_m$, $\lim \int \underline{f}_n \leq \liminf \int f_n$

Thm. If $f, g : \mathbb{R} \to [0,\infty)$ are measurable, then $\int f+g = \int f + \int g$

pf: Recall:

- Mesomorphism (measurability preserving maps)

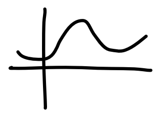
  $f : \mathbb{R}^n \xrightarrow{1-1} \mathbb{R}^n$

- Mesometry (measure preserving maps (bijection))

  $f : \mathbb{R}^n \to \mathbb{R}^n$

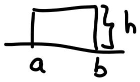
If $f : \mathbb{R} \to \mathbb{R}$, we define

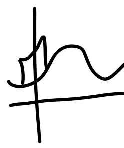
$T_f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x, y+f(x))$

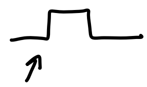
e.g.  f , T_f : $\underline{\quad y=0 \quad}$ $\to$  $y=f(x)$

Note T_f has inverse T_f^{-1}

$\boxed{\text{Thm}}$ (35) T_f is an mesometry if $f:\mathbb{R}\to[0,\infty)$ measurable

Let $R=$  $\subset \mathbb{R}^2$

$T_f R =$  f

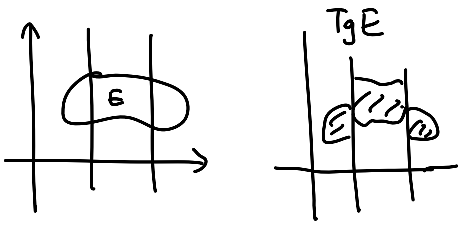


W.T.S.: $T_f R = u(f+h\mathbb{1}_{[a,b]}) - u(f)$, let $g = h\mathbb{1}_{[a,b]}$

suffice to show $u(f+h\mathbb{1}_{[a,b]})$ measurable

$$\hat{u}(f+g) = T_f \hat{U}_g = T_g \hat{U}_f$$

• Claim: T_g preserves measurability.



Translation

• Assume $f = f\mathbb{1}_{[a,b]}$, then

$$u(f) \sqcup T_f R = T_g(U(f)) \sqcup R$$

$$\Rightarrow m(U(f)) + m(T_f R) = m(T_g U_f) + m(R)$$

$$= m(U_f) + m(R) \quad (T_g \text{ is mesometry})$$

$$\Rightarrow m(T_f R) = m(R)$$

• Claim: T_f never increase outer measure.

pf: $\forall A \subset \mathbb{R}^k$, $\forall \varepsilon>0$, $\exists$ boxes $\{R_i\}$ s.t. $\sum m(R_i) = m^*(A)+\varepsilon$

$$\Rightarrow m^*(T_f A) \le \sum_{i=1}^{\infty} m(T_f R_i) \quad (\text{since } T_f A \subset \bigcup_i T_f R_i$$

$$= \sum_{i=1}^{\infty} m(R_i)$$

$$\le m^*(A)+\varepsilon$$

$\Rightarrow m^*(T_f A) = m^*(A)$

Since $T_{-f} = \varphi \cdot T_f \cdot \varphi$, $\varphi : (x,y) \mapsto (x,-y)$

$\Rightarrow \varphi$ is mesometry

$\Rightarrow T_{-f}$ preserves measurability

$\Rightarrow m^*(T_{-f}(A)) \leq m^*(A)$

Also, $m^*(A) = m^*(T_f T_{-f} A)$

$\leq m^*(T_{-f} A)$

$\leq m^*(A)$

$\Rightarrow T_f$ preserves outer measure & measurability

Thm. 35 : $m(\mathcal{U}(f+g)) = m(T_f \mathcal{U}_g \cup \mathcal{U}_f)$

$= m(T_f \mathcal{U}_g) + m(\mathcal{U}_f)$

$= m(\mathcal{U}_g) + m(\mathcal{U}_f)$

$\boxed{\text{Corollary}}$: If $\{f_n\}$ is a seq. of integrable fxn., then

$$\int \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int f_k$$

Pf: Let $F_k = \sum_{i=1}^{k} f_i$, $F_k \nearrow F$, $\int F_k \nearrow \int F$

$$\Rightarrow \sum_{k=1}^{n} \int f_k = \int \sum_{k=1}^{n} f_k$$