- Indicator Fxns. (Characteristic Fxn.)
  - $E \subset \mathbb{R}^n$ measurable
  - $\mathbb{1}_E(x) = \begin{cases} 1 & x \in E \\ 0 & \text{else} \end{cases}$

- <u>Simple fxns.</u> are finite lin. combo. of indicator

$$f(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{E_i}(x), \quad E_i \subset \mathbb{R}^n, \; c_i \in \mathbb{R}$$

$\Rightarrow \int \mathbb{1}_E = m(E)$

$\int \Sigma c_i \mathbb{1}_E = \Sigma c_i m(E_i)$

$\Rightarrow$ For measurable fxn. (nonnegative)

① Find simple fxn. $f_n : \mathbb{R}^n \to [0,\infty)$ s.t. $f_n \nearrow f$ ptwise.

② Define $\int f = \lim_{n\to\infty} \int f_n$

Tao 7.5: Measurable Fxn.

- $f : \mathbb{R}^n \to \mathbb{R}$ measurable if $\forall V \subset \mathbb{R}$ open,

  $f^{-1}(V)$ is measurable.

* More generally, say $\Omega \subset \mathbb{R}^n$ a measurable set. $f : \Omega \to \mathbb{R}$, then f is measurable if $f^{-1}(\text{open})$ is measurable in $\mathbb{R}^n$.

$\boxed{\text{Lemma}}$: If $\Omega \subset \mathbb{R}^n$ measurable, $f : \Omega \to \mathbb{R}$ con't. Then f is measurable.

$\boxed{\text{Pf}}$ $\forall V \subset \mathbb{R}$ open, we have $f^{-1}(V) \subset \Omega$ is open in Ω,

? i.e. $f^{-1}(V) = W \cap \Omega$, where $W \subset \mathbb{R}^n$ is open. Since W and Ω are measurable, $W \cap \Omega$ are measurable.

$\boxed{\text{Lemma}}$ If $f: \Omega \to \mathbb{R}^k$ measurable, and $g: \mathbb{R}^k \to \mathbb{R}^L$ con't, then $g \circ f$ is measurable.

$\boxed{\text{Pf}}$ $\Omega \longrightarrow \mathbb{R}^k \longrightarrow \mathbb{R}^L$

$\underset{\text{measurable}}{f^{-1}(g^{-1}(V))} \longrightarrow \underset{\text{open}}{g^{-1}(V)} \longrightarrow \underset{\text{open}}{V}$

$\boxed{\text{Lemma}}$ A fxn. $f = (f_1, \cdots, f_n): \mathbb{R}^k \to \mathbb{R}^n$ is measurable

$\iff f_1, \cdots, f_n \ \mathbb{R}^k \to \mathbb{R}$ are measurable

$\boxed{\text{Pf}}$ ($\Rightarrow$) $f_i = \pi_i \circ f$, $\pi_i: \mathbb{R}^n \to \mathbb{R}$ project to i-th component

$\Rightarrow f_i$ is meas. $\forall i$

($\Leftarrow$) To prove f is meas., suffice to check if $\overbrace{f^{-1}(\text{open box})}^{(1)}$ is meas.

$\boxed{\text{Pf}}$ (1) Any open set in $\mathbb{R}^n$ is a countable union of open boxes.

$f^{-1}((a_1, b_1) \times \cdots \times (a_n, b_n))$

$= \bigcap_{i=1}^{n} f_i^{-1}((a_i, b_i))$

$= \bigcap_{i=1}^{n} (\text{measurable set})$

$= \text{meas.}$

$\boxed{\text{Cor}}$ • If f is measurable $\mathbb{R} \to \mathbb{R}$, then $|f|$ is measurable.

$\boxed{\text{Pf}}$ $\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{|\cdot|} \mathbb{R}$

• If f_1, f_2 are meas. $\mathbb{R} \to \mathbb{R}$, then $f_1 + f_2$ is meas.

$\boxed{\text{Pf}}$ $\mathbb{R} \xrightarrow{(f_1, f_2)} \mathbb{R}^2 \xrightarrow{\cdot + \cdot} \mathbb{R}$

$x \longmapsto (f_1(x), f_2(x)) \longmapsto f_1(x) + f_2(x)$

Reconciling Pugh & Tao's def. of measurable fxn.

Let $f : \mathbb{R} \to [0,\infty)$

$U(f)$ measurable $\overset{?}{\iff}$ $\forall V$ open in $\mathbb{R}$, $f^{-1}(V)$ meas.

$\iff \forall (a,\infty), a \geq 0, f^{-1}((a,\infty))$ is meas.

$\boxed{\text{Pf}}$ • $f^{-1}((-\infty,a]) = [f^{-1}((a,\infty))]^c$ meas.

• $a<b$, then $f^{-1}([a,b]) = f^{-1}((a,\infty)\setminus(b,\infty))$

$= f^{-1}((a,\infty)) \setminus f^{-1}((b,\infty))$

$=$ meas.

• $\{b\} = \bigcap_{n=1}^{\infty} (b-\frac{1}{n}, b] \Rightarrow f^{-1}(\{b\})$ meas.

or $(a,b) = \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}]$

$\Rightarrow f^{-1}((a,b)) = \bigcup_n f^{-1}((a, b-\frac{1}{n}))$

* Warning : • If $E \subset \mathbb{R}^2$ meas., it doesn't mean $\pi_1(E)$ meas.

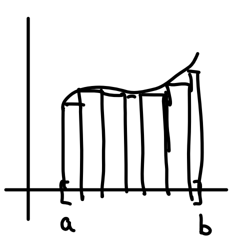
$(\pi_1 : \mathbb{R}^2 \to \mathbb{R})$

• If $f : \mathbb{R} \to \mathbb{R}^2$ con't. fxn., $E \subset \mathbb{R}^2$ meas., $f^{-1}(E) \subset \mathbb{R}$ may not be meas.

• Integration

Riemann Integral :

$f : [a,b] \to [0,\infty)$ con't.



boxes are the basic object

Lebesgue Integral

• $E \subset \mathbb{R}$ be a meas. set

$\mathbb{1}_E(x)$

• Def: $f : \mathbb{R} \to \mathbb{R}$ is a meas. fxn. and $f(\mathbb{R})$ is a finite set, then f is called a simple fxn.

$\boxed{\text{Lemma}}$ If f is a simple fxn., then $\exists$ $E_1, \dots, E_n$ disjoint meas. subsets of $\mathbb{R}$,

$c_1, \dots, c_n \in \mathbb{R}$, s.t. $f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}(x)$

$\boxed{\text{Pf}}$ Let $f(\mathbb{R}) = \{c_1, \dots, c_n\}$. $\forall c_i$, choose $\varepsilon_i > 0$ small enough s.t.

$(c_i - \varepsilon_i, c_i + \varepsilon_i) \cap f(\mathbb{R}) = \{c_i\}$, then

$f^{-1}(c_i) = f^{-1}((c_i - \varepsilon_i, c_i + \varepsilon_i)) = E_i$. Thus $f(x) = \sum c_i \mathbb{1}_{E_i}(x)$

Prop: The set of simple fxns. form a vector space. I.e.,

- $\forall c \in \mathbb{R}$, $\forall f$ simple fxn., cf is simple
- $\forall f, g$ simple fxn., $f+g$ is simple

Let f be a simple fxn., $f : \mathbb{R} \to [0, \infty)$, then

$$\int f = \sum_{\substack{\lambda \in f(\mathbb{R}) \\ \lambda > 0}} \overset{\text{height}}{\lambda} \cdot \underbrace{m(f^{-1}(\lambda))}_{\text{base width}}$$

$\boxed{\text{Lemma}}$ ① $\forall c > 0$, $f : \mathbb{R} \to [0, \infty)$ simple, $\int c \cdot f = c \int f$

② If $f, g : \mathbb{R} \to [0, \infty)$ simple, $\int f + g = \int f + \int g$

$\text{supp}(f) = \{x \mid f(x) \neq 0\}$

③ Let $f : \mathbb{R} \to [0, \infty)$ be meas., then $\exists$ f_n seq. of non-neg. simple fxn. of bounded support, s.t. $f_n \to f$ ptwise.

$\boxed{\text{Pf}}$ ② Suffice to show, $\forall E_i$ meas. (may not be disjoint)

$$\int \sum c_i \mathbb{1}_{E_i}(x) = \sum c_i \, m(E_i)$$

Let $E_{ij} = E_i \cap E_j$, $c_{ij} = c_i + c_j$, then

$$\sum c_i \mathbb{1}_{E_i}(x) = \sum_{1 \le i \le j \le n} c_{ij} \mathbb{1}_{E_{ij}}(x)$$

?

③ Let $f_n(x) = \sup\{\frac{j}{2^n} \mid \frac{j}{2^n} \le \min(f(x), 2^n), j \in \mathbb{Z}_{\ge 0}\}$ $f_n(x) = \mathbb{1}_{[-n,n]}(x)$

(1) $f_n(\mathbb{R}) \subset \{\frac{j}{2^n} : \frac{j}{2^n} \le 2^n, j \in \mathbb{Z}_{\ge 0}\}$ finite set

(2) $f_n(x) = \frac{j}{2^n}$, $\frac{j}{2^n} \ne 2^n \Leftrightarrow \frac{j}{2^n} \le f(x) < \frac{j+1}{2^n}$

$\Leftrightarrow x \in f^{-1}([\frac{j}{2^n}, \frac{j+1}{2^n}])$

$f_n(x) = 2^n \Leftrightarrow f(x) \ge 2^n$

$\Leftrightarrow x \in f^{-1}([2^n, \infty))$

Prop : $f : \mathbb{R} \to [0,\infty)$ is meas. $\Leftrightarrow f$ is a ptwise limit of a seq. of simple fxns.

$\boxed{\text{Pf}}$ $(\Rightarrow)$ by above construction

$(\Leftarrow)$ Show: If f_n is a seq. of non-neg. meas. fxn., $f_n \to f$ ptwise., then f is meas.

$\underline{f_n} \to f$ $\overline{f_n} \to f$ $f_n \to f$ ptwise.

Check: $f^{-1}([a,\infty)) = \bigcap_{n=1}^{\infty} \underline{f_n}^{-1}([a,\infty))$