Measure Theory Notes

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Contents

1 Jordan & Lebesgue measure

Definition 1. *An* ***interval*** *is a subset of* $\mathbb{R}$ *of form*

$$[a,b] \coloneqq \{x \in \mathbb{R} : a \leq x \leq b\},$$

$$[a,b) \coloneqq \{x \in \mathbb{R} : a \leq x < b\},$$

$$(a,b] \coloneqq \{x \in \mathbb{R} : a < x \leq b\},$$

$$(a,b) \coloneqq \{x \in \mathbb{R} : a < x < b\}.$$

We define ***length*** *of such an interval* I *as* $|I| \coloneqq b-a$. *A* ***box*** *in* $\mathbb{R}^d$ *is a Cartesian product* $B \coloneqq I_1 \times ... \times I_d$ *of* d *interval* $I_1, ..., I_d$. *We define the* ***volume*** $|B|$ *of such a box as* $|B| \coloneqq |I_1| \times ... \times |I_d|$.

Definition 2. *An* ***elementary set*** *is a subset of* $\mathbb{R}^d$ *which is the union of finitely many boxes.*

Lemma 1 (Tao MT, 1.1.2, Elementary measure)**.** *Let* $E \subset \mathbb{R}^d$ *be an elementary set.*

(i) E *can be expressed as the finite union of disjoint boxes.*

(ii) Elementary measure $m(E)$ *is defined to be the sum of these disjoint boxes, and is independent of partition.*

Proof idea.

(i) Let each endpoint of the elementary set boxes define a new mesh.

(ii) Discretisation or mutual refinement of any two partitions of E.

□

Note, measure defined by discretisation can fail to obey desired properties. Consider, for instance, $\mathbb{Q} \cap [0,1]$ and $\mathbb{Q} \cap ([0,1] + \sqrt{2})$.

Exercise 1 (Tao MT 1.1.3)**.** *uniqueness of elementary measure. Idea: similar to our exercise with normalization property on the unit cube, but with a linear scaling of c given unit cube of measure c.*

Definition 3. *Let* $E \subset \mathbb{R}$ *be a bounded set. The* ***Jordan inner measure*** *of* E *is defined as*

$$m_{*,(J)} \coloneqq \underset{A \subset E, A\ elementary}{sup} m(A).$$

*The **Jordan outer measure** of E is defined as*

$$m^{*,(J)} := \inf_{E \subset B, B \ elementary} m(B).$$

E is Jordan measurable if $m_{*,(J)} = m^{*,(J)}$.

An example of the limitation of Jordan measure (Tao MT, Exercise 1.2.1): Given a bijection $f : \mathbb{N} \to \mathbb{Q} \cap [0,1]$, a sequence of all the rationals (q_i) and fixed $N \in \mathbb{N}$, let $((q_n)_{n \in [N]})_N$ be a s the collection of sets rationals given by $q_n = \{f(n) : n \in [N]\}$, and let $A_N = \{q_n : n \in [N]\}$. Then for any M, $m(\cup_{N=1}^{M} A_N) = 0$, but $m(\bigcup_{N=1}^{\infty} A_N) = 1$.

On a related note (Tao MT, Exercise 1.2.2), if g_N is the indicator of A_N on $[0,1] \subset \mathbb{R}$, then (g_N) is a sequence of uniformly bounded Riemann integrable functions $g_N : [0,1] \to \mathbb{R}$ that converge pointwise to a bounded function $f : [0,1] \to \mathbb{R}$ that is not Riemann integrable. We can replace pointwise convergence with uniform convergence and find a similarly problematic sequence of functions $h_N : [0,1] \to \mathbb{R}$, where we let $h_0 : [0,1] \to \{0\}$, and

$$h_N(x) = \begin{cases} 1/N & \text{if}\, h_N(x) \in \mathbb{Q} \cap A_N \setminus A_{N-1} : \\ h_{N-1}(x) & \text{if}\, h_N(x) \in A_{N-1} \\ 0 & \text{if}\, x \notin \mathbb{Q}. \end{cases}$$

Definition 4. *For $E \subset \mathbb{R}^d$, we define the **Lebesgue outer measure** $m^*(E)$ of E as*

$$m^*(E) := \inf_{\cup_{n=1}^{\infty} B_n \supset E; B_1, B_2, \ldots boxes} \sum_{n=1}^{\infty} |B_n|.$$

Note,

- $m^*(E) \leq m^{*,(J)}$, as any finite set can be made countable by adding infinitely many empty boxes. Also, when covering a countable set, we can use degenerate boxes, or boxes with arbitrarily small, decreasing side length.
- Unbounded sets can have Lebesgue outer measure zero, but not Jordan outer measure zero.

Definition 5. *A set $E \subset \mathbb{R}$ is said to be **Lebesgue measurable** if, for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^d$ containing E such that $m^*(U \setminus E) \leq \epsilon$. If E is Lebesgue measurable, then we refer to $m(E) := m^*(E)$ as the Lebesgue measure of E.*

Exercise 2 (Tao MT, 1.2.3)**.** *Outer measure axioms*

(i) *(Empty set)* $m^*\emptyset = 0$.

(ii) *(Monotonicity) If* $E \subset F \subset \mathbb{R}$*, then* $m^*(E) \leq m^*(F)$.

(iii) *(Countable subadditivity) if* $E_1, E_2, ... \subset \mathbb{R}$ *is a countable sequence of sets, then* $m^*(\cup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

Note that we use of Tonelli's theorem for series in proving (iii).

Lemma 2 (Finite additivity for separated sets)**.** *Let* $E, F \subset \mathbb{R}^d$ *be such that* $dist(E, F) > 0$*, where*

$$dist(E, F) := (inf)\{|x - y| : x \in E, y \in F\}$$

is the (Euclidean) distance between E and F. Then

$$m^*(E \cup F) = m^*(E) + m^*(F).$$

Lemma 3. *Let* $E, F \subset \mathbb{R}^d$ *be disjoint closed sets, at least one of them compact. Then* $dist(E, F) > 0$.

Lemma 4. *Let* E *be an elementary set. Then the Lebesgue measure* $m^*(E)$ *of* E *is equal to the elementary measure* $m(E)$ *of* E*, that is,* $m^*(E) = m(E)$.

Proof. We have previously noted that $m^*(E) \leq m(E)$. We prove the lemma first for the case that E is closed. Since elementary sets are bounded, E is closed and bounded, and by the Heine-Borel Theorem is compact. Fix arbitrary $\epsilon > 0$. Then there exists a family of boxes $(B_n)_{n \in N}$, N countable, that cover E,

$$E \subset \cup_{n=1}^{\infty} B_n,$$

and such that

$$\sum_{n=1}^{\infty} |B_n| \leq m^*(E) + \epsilon.$$

The boxes B_N are not necessarily open, but we can enlarge the boxes to be open and use Heine-Borel. Replace each box B_n by $B'_n \supset B_n$ such that

$$|B'_n| \leq |B_n| + \epsilon/2^n,$$

so that $(B'_n)_{n \in N}$ covers E, and

$$\sum_{n=1}^{N} |B'_n| \leq \sum_{n=1}^{\infty} (|B_n| + \epsilon/2^n) = (\sum_{n=1}^{\infty} |B_n|) + \epsilon \leq m^*(E) + 2\epsilon.$$

Applying Heine-Borel to E and the open cover $(B'_n)_{n\in N}$, we have

$$E \subset \cup_{n=1}^{N} B'_n$$

for some finite N. By subadditivity of elementary measure, we conclude

$$m(E) \leq \cup_{n=1}^{N} B'_n$$

and thus

$$m(E) \leq m^*(E) + 2\epsilon.$$

Letting $\epsilon \to 0$, the lemma follows for closed E.

In the case that the elementary set E is not closed, we may write E as the finite union $Q_1 \cup ... \cup Q_k$ of disjoint boxes of any form. For every $\epsilon > 0$ and every $1 \leq j \leq k$, there exists a closed sub-box Q'_j of Q_j such that

$$|Q'_j| \geq |Q_j| - \epsilon/k,$$

so E contains the finite union $Q'_1 \cup ... \cup Q'_k$ of disjoint closed boxes, a closed elementary set. By finite additivity of elementary measure , and the prior result on closed elementary sets,

$$m^*(Q'_1 \cup ... \cup Q'_k) = m(Q'_1 \cup ... \cup Q'_k)$$

$$= m(Q'_1) + ... + m(Q'_k) \geq m(Q_1) + ... + m(Q_k) - \epsilon$$

$$= m(E) - \epsilon.$$

Then by monotonicity of Lebesgue outer measure we have

$$m^*(E) \geq m(E) - \epsilon$$

and the lemma follows. □

Definition 6. *Two boxes are* ***almost disjoint*** *if their interiors are disjoint.*

Recalling that a box has the same elementary measure as its interior (following from definition and partition invariance), we have finite additivity

$$m(B_1 \cup ... \cup B_k) = |B_1| + ... + |B_k|$$

for almost disjoint boxes, as such boxes have disjoint interior. This with the previous lemma lets us prove:

Lemma 5. *Let $E = \cup_{n=1}^{\infty} B_n$ be a countable union of almost disjoint boxes $B_1, B_2, \ldots$. Then*

$$m^*(E) = \sum_{n=1}^{\infty} |B_n|.$$

Proof. Countable addtivity and the previous lemma give

$$m^*(E) \leq \sum_{n=1}^{\infty} m^*(B_n) = \sum_{n=1}^{\infty} |B_n|,$$

and we next show that

$$\sum_{n=1}^{\infty} |B_n| \leq m^*(E).$$

For any $N \in \mathbb{N}$, we see that

$$E \supset B_1 \cup \ldots \cup B_N,$$

so by monotonicity and the previous lemma,

$$m^*(E) \geq m^*(B_1 \cup \ldots \cup B_N) = m(B_1 \cup \ldots \cup B_N).$$

By finite additivity for almost disjoint boxes, we have

$$\sum_{n=1}^{N} |B_n| \leq m^*(E),$$

and letting $N \to \infty$, the result follows. □

Note, we can see by this lemma that R^d has infinite outer measure.

Lemma 6. *Any open set $E \subset \mathbb{R}^d$ can be expressed as the countable union of almost disjoint boxes.*

Proof idea. Use the dyadic mesh of cubes of form

$$Q = \prod_{k=1}^{d} \left[\frac{i_k}{2^n}, \frac{i_k + 1}{2^n} \right]$$

for appropriately chosen n, d, incrementing n as needed to cover all of E. □

Lemma 7 (Outer Regularity). *Let $E \subset \mathbb{R}^d$ be an arbitrary set. Then*

$$m^*(E) = \inf_{E \subset U,\ U\, open} m^*(U).$$

Proof sketch. Monotonicity of Lebesgue measure gives

$$m^*(E) \leq \inf_{E \subset U,\ U \text{open}} m^*(U).$$

To see that

$$\inf_{E \subset U,\ U \text{open}} m^*(U) \leq (E),$$

when $m^*(E)$ is not infinite, we acknowledge a countable cover $(B_j)_{j \in J}$ of E, such that $\sum_{n=1}^{\infty} |B_n| \leq m^*(E) + \epsilon$, and cover each B_n with a box B'_n such that $|B_n| \leq |B'_n| + \epsilon/2^n$, giving a new cover of E. Taking the union of these boxes, countable subadditivity gives

$$m^*(\cup_{n=1}^{\infty} B'_n) \leq m^*(E) + 2\epsilon$$

. ∎

Lemma 8 (Existence of Lebesgue measurable sets).

(i) Every open set is Lebesgue measurable.

(ii) Every closed set is Lebesgue measurable.

(iii) Every set of Lebesgue measure zero is measurable.

(iv) The empty set $\emptyset$ is Lebesgue measurable.

(v) if $E \subset R^d$ is Lebesgue measurable, then so is its complement $R^d \setminus E$.

(vi) The countable union of Lebesgue measurable sets is measurable.

(vii) The countable intersection of Lebesgue measurable sets is Lebesgue measurable.

Note, properties (iv), (v), and (vi) show that the collection of Lebesgue measurable subsets of R^d form a σ-algebra. It contains the limits of every contained monotone sequence.

Exercise 3 (Criteria for measurability).

Exercise 4 (Criteria for finite measure).

2 The Lebesgue integral

The Lebesgue theory extends the Riemann theory. Every Jordan measurable set is Lebesgue measurable, and every Riemann integrable function is Lebesgue measurable. Conversely, the Lebesgue theory can be approximated by the Riemann theory. The Lebesgue theory is complete in various ways (explain).

Definition 7. *A complex-valued* ***simple function*** $f : \mathbb{R}^d \to \mathbb{C}$ *is a finite linear combination*

$$f = c_1 1_{E_1} + ... + c_k 1_{E_k}$$

of indicator functions 1_{E_i} *of lebesgue measurable sets* $E_i \subset \mathbb{R}^d$ *for* $i = 1, ..., k$*, where* $k \geq 0$ *is a natural number and* $c_1, ..., c_k \in \mathbb{C}$*. An* ***unsigned simple function*** $f : \mathbb{R}^d \to [0, +\infty]$*, is defined similarly, but such that* $c_i \in [0, +\infty]$ *rather than* $\mathbb{C}$*.*

It follows from this definition that $\mathrm{Simp}(\mathbb{R}^d)$ is a vector space of complex values functions, closed under pointwise product and conjugation. $\mathrm{Simp}(\mathbb{R}^d)$ is a commutative *-algebra. The space $\mathrm{Simp}^+(\mathbb{R}^d)$ of unsigned simple functions is a $[0, inf]$-module, closed under addition, and scalar multiplication by elements in $[0, +\inf]$.

Definition 8. *If* $f = c_1 1E_1 + ... + c_k 1E_k$ *is an unsigned simple function, we define the integral*

$$Simp \int_{\mathbb{R}^d} f(x)dx \coloneqq c_1 m(E_1) + ... + c_k m(E_k),$$

which takes values in $[0, +\infty]$*.*

Lemma 9 (Well-definedness of simple integral)**.** *Let* $k, k' > 0$ *be natural numbers,* $c_1, ..., c_k, c'_1, ..., c'_{k'} \in [0, \infty]$*, and let* $E_1, ..., E_k, E'_1, ..., E'_{k'} \subset \mathbb{R}^d$ *be Lebesgue measurable sets such that the identity*

$$c_1, ..., c_k = c'_1, ..., c'_{k'}$$

hold identically on R_d*. Then*

$$c_1 m(E_1), ..., c_k m(E_k) = c'_1 m(E'_1), ..., c'_{k'} m(E'_{k'}).$$

Definition 9. *A property* $P(x)$ *of a point* $x \in \mathbb{R}^d$ *is said to hold* ***(Lebesgue) almost everywhere*** *in* $\mathbb{R}^d$*, or for* ***(Lebesgue) almost every point*** $x \in R^d$*, if the set of* $x \in \mathbb{R}^d$ *for which* $P(x)$ *fails has Lebesgue measure zero.*

Two function $f, g : \mathbb{R}^d \to Z$ into an arbitrary range Z are said to ***agree almost everywhere*** *if $f(x) = g(x)$ for almost every $x \in R^d$.*

The ***support*** *of a function $f : \mathbb{R}^d \to \mathbb{C}$ or $f : \mathbb{R}^d \to [0, +\infty]$ is defined to be the set $x \in \mathbb{R}^d : f(x) \neq 0$.*

Note, "almost every" behaves like the quantifier "for every" when dealing in a countable number of properties.

Definition 10. *A complex-valued simple function $f : \mathbb{R}^d \to \mathbb{C}$ is said to be* ***absolutely integrable*** *if*

$$Simp \int_{\mathbb{R}^d} |f(x)| dx < \infty.$$

If f is absolutely integrable, the integral is defined for real signed f by

$$Simp \int_{\mathbb{R}^d} f(x) dx := Simp \int_{\mathbb{R}^d} f_+(x) dx - Simp \int_{R^d} f_-(x) dx$$

where $f_+(x) := \max(f(x), 0)$ and $f_-(x) := \max(-f(x), 0)$. These functions are pointwise dominateed by $|f|$ and thus have a finite integral. For complex-valued f we have

$$Simp \int_{R^d} f(x) dx := Simp \int_{R^d} Ref(x) dx + i Simp \int_{R^d} Imf(x) dx.$$

Definition 11. *An unsigned function $f : \mathbb{R}^d \to [0, +\infty]$ is* ***unsigned Lebesgue measurable****, or* ***measurable****, if it is the pointwise limit of unsigned simple functions, that is, if there exists a sequence $f_1, f_2, f_3, ... : \mathbb{R}^d \to [0, +\infty]$ of unsigned simple functions such that $f_n(x) \to f(x)$ for every $x \in \mathbb{R}^d$.*

Note, simple functions that agree almost everywhere are *noise tolerant* in the sense that there can be "noise" or "error" in a function on a null set without affecting the final integral values. We can can also integrate functions that are not defined everywhere, but are defined almost everywhere on $\mathbb{R}^d$. For example, consider the function $\sin(x)/x$ on $\mathbb{R}$.

Lemma 10. *Let $f : R^d \to [0, +\infty]$ be an unsigned function. Then the following are equivalent:*

(i) f is unsigned Lebesgue measurable.

(ii) f is the pointwise limit of unsigned simple functions f_n.

(iii) f is the pointwise almost everywhere limit of unsigned simple functions f_n.

(iv) *f is the supremum $f(x) = \sup_n f_n(x)$ of an increasing sequence $0 \leq f_1 \leq f_2 \leq f_3 \leq \ldots$ of unsigned simple functions f_n, each bounded with finite measure support.*

(v) *For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) > \lambda\}$ is Lebesgue measurable.*

(vi) *For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) \geq \lambda\}$ is Lebesgue measurable.*

(vii) *For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) < \lambda\}$ is Lebesgue measurable.*

(viii) *For every $\lambda \in [0, +\infty]$, the set $\{x \in \mathbb{R}^d : f(x) \leq \lambda\}$ is Lebesgue measurable.*

(ix) *For every interval $I \subset [0, +\infty)$, the set $f^{-1}(I) \coloneqq \{x \in \mathbb{R}^d : f(x) \in I\}$ is Lebesgue measurable.*

(x) *For every (relatively) open set $U \subset [0, +\infty)$, the set $f^{-1}(U) \coloneqq \{x \in \mathbb{R}^d : f(x) \in U\}$ is Lebesgue measurable.*

(xi) *For every (relatively) closed set $K \subset [0, +\infty)$, the set $f^{-1}(K) \coloneqq \{x \in \mathbb{R}^d : f(x) \in K\}$ is Lebesgue measurable.*

Proof (of some equivalencies). (i) and (ii) are equivalent by definition, and (iii) follows from (ii). Every monotone sequence in $[0, +\infty]$ converges, so (iv) implies (ii). To see that (iii) implies (v), observe that if f is the pointwise almost everywhere limit of f_n, then for almost every $x \in \mathbb{R}^d$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) = \inf_{N>0} \sup_{n\geq N} f_n(x).$$

Then for any λ, up to null sets,

$$\{x \in \mathbb{R}^d : f(x) > \lambda\} = \bigcup_{M>0} \bigcap_{N>0} \{x \in \mathbb{R}^d : \sup_{n\geq N} f_n(x) > \lambda + \frac{1}{M}\}$$

$$= \bigcup_{M>0} \bigcap_{N>0} \bigcup_{n>N} \{x \in \mathbb{R}^d : f_n x > \lambda + \frac{1}{M}\}.$$

Since each f_n is an unsigned simple function, each set $\{x \in \mathbb{R}^d : f_n x > \lambda + \frac{1}{M}\}$ is obtained by finite intersection and finite union of measurable sets, and is therefore measurable.

(iv) and (v) are equivalent, as

$$\{x \in \mathbb{R}^d : f(x) \geq \lambda\} = \bigcap_{\lambda' \in \mathbb{Q}^+ : \lambda' < \lambda} \{x \in \mathbb{R}^d : f(x) > \lambda'\},$$

$$\{x \in \mathbb{R}^d : f(x) > \lambda\} = \bigcup_{\lambda' \in \mathbb{Q}^+ : \lambda' > \lambda} \{x \in R^d : f(x) \geq \lambda'\}.$$

(v)-(xi) implies (v): Let f obey (v)-(ix). Then for each $n \in \mathbb{N}^+$, let $f_n(x)$ be defined as the largset integer multiple of 2^{-n} that is less than or equal to $\min(f(x), n)$, with $|x| \leq n$, and $f_n(x) \coloneqq 0$ for $|x| > n$. The $f_n : [0, \infty]$ are increasing, and $\sup f_n = f$. Each f_n attains only finitely many values, and for each such value c, the set $f_n^{-1}(c)$ has form $f^{-1}(I_c) \cap \{x \in \mathbb{R}^d : |x| \leq n\}$, and is thus measurable. By construction, f_n is a simple function that is bounded with finite measure support. □

Exercise 5 (Tao MT, 1.3.3)**.** *There are plenty of measurable functions.*

(i) *Continuous functions $R^d \to [0, +\infty]$ are measurable.*

(ii) *Unsigned simple functions are measurable.*

(iii) *The supremum, infimum, limit superior, and limit inferior of unsigned measurable functions are measurable.*

(iv) *An unsigned function that is equal almost everywhere to an unsigned measurable function is measurable.*

(v) *If a sequence of unsigned measurable functions converges pointwise almost everywhere to an unsigned limit f, then f is also measurable.*

(vi) *If $f : R^d \to [0, \infty]$ is measurable and $\phi : [0, +\infty]$ is continuous, then $\phi \circ f : \mathbb{R}^d \to [0, \infty]$ is measurable.*

(vi) *If f, g are unsigned measurable functions, then $f+g$ and fg are measurable.*

Proof. (i): Let $U \subset [0, +\infty]$ be and open set, and f be a continuous function $f : \mathbb{R}^d \to [0, +\infty]$. Because f is continuous, $f^{-1}(U)$ is open, and therefore Lebesgue measurable. By Lemma 1.3.9 (ix), f is measurable.

(ii): This follows from Lemma 1.3.9 (iv).

(iii): Follows from fact that

$$\{x \in \mathbb{R}^d : \sup_{n \in \mathbb{N}} f_n(x) \geq \lambda\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R}^d : f_n(x) \geq \lambda\},$$

the right hand side being the countable union of measurable sets.

(iv): This function satisfies Lemma 1.3.9 (iii)

(v) Approximating each f_n with simple functions gives this fact, as does taking the supremum, infimum, etc. of the function, and using (iii).

(vi) Follows from taking inverse of an open set and the resulting open preimage.

(v) Taking pointwise sums and products of measurable function, we have pointwise convergence, and thus fg and $f+g$ are pointwise convergent measurable functions, since the sum and product of finitely many simple functions is simple. $\square$

Note, if we extend a Riemann integrable function $f : [a,b] \to \mathbb{R}$ by defining $f(x) = 0$ for $x \notin [a,b]$, then f is measurable.

Definition 12. *An almost everywhere defined complex-valued function $f : \mathbb{R}^d \to \mathbb{C}$ is **Lebesgue measurable** or **measurable** if it is the pointwise almost everywhere limit of complex-valued simple functions.*

Definition 13. *Let $f : \mathbb{R}^d \to [0, +\infty]$ be an unsigned function, not necessarily measurable. We define the lower Lebesgue integral $\underline{\int_{R^d}} f(x)dx$ to be the quantity*

$$\underline{\int_{R^d}} f(x)dx := \sup_{0 \leq g \leq f;\ g\ simple} Simp \int_{R^d} g(x)dx$$

where g ranges over all unsigned simple functions $g : R^d \to [0, +\infty]$ that are pointwise bounded by f.

We define the upper Lebesgue integral $\overline{\int_{R^d}} f(x)dx$ to be the quantity

$$\overline{\int_{R^d}} f(x)dx := \inf_{h \geq f;\ h\ simple} Simp \int_{R^d} h(x)dx$$

where g ranges over all unsigned simple functions $h : R^d \to [0, +\infty]$. We will mostly use the lower Lebesgue integral. Both take values in $[0, +\infty]$, and $\underline{\int_{R^d}} f(x)dx \leq \overline{\int_{R^d}} f(x)dx$.

Exercise 6 (Basic properties of the lower Lebesgue integral)**.** *Let $f, g : R^d/to[0, +\infty]$ be unsigned functions (not necessarily measurable).*

(i) (Compatibility with the simple integral)If f is simple then

$$\underline{\int_{R^d}} f(x)dx = \overline{\int_{R^d}} f(x)dx = Simp \int_{R^d} f(x)$$

(ii) (Monotonicity) If $f \leq g$ pointwise almost everywhere, then we have

$$\underline{\int_{R^d}} f(x)dx \leq \underline{\int_{R^d}} g(x)$$

and

$$\overline{\int_{R^d}} f(x)dx \leq \overline{\int_{R^d}} g(x)dx$$

(iii) *(Homogeneity) If $c \in [0, +\infty)$, then*

$$\underline{\int}_{R^d} f(x)dx = c\underline{\int}_{R^d} f(x)dx$$

(iv) *(Equivalence) If f, g agree almost everywhere, then*

$$\underline{\int}_{R^d} f(x)dx \leq \underline{\int}_{R^d} g(x)dx$$

and

$$\overline{\int}_{R^d} f(x)dx \leq \overline{\int}_{R^d} g(x)dx$$

(v) *(Superadditivity)*

$$\underline{\int}_{R^d} f(x) + g(x)dx \geq \underline{\int}_{R^d} f(x)dx + \underline{\int}_{R^d} g(x)dx$$

(vi) *(Subadditivity of upper integral)*

$$\overline{\int}_{R^d} f(x) + g(x)dx \leq \overline{\int}_{R^d} f(x)dx + \overline{\int}_{R^d} g(x)dx$$

(vii) *(Divisibility) For any measurable set E,*

$$\underline{\int}_{R^d} f(x)dx = \underline{\int}_{R^d} f(x)1_E(x) + \underline{\int}_{R^d} f(x)1_{\mathbb{R}^d\setminus} dx$$

(viii) *(Horizontal truncation) As $n \to \infty$,*

$$\underline{\int}_{R^d} \min(f(x), n)dx \to \underline{\int}_{R^d} f(x)dx$$

(ix) *(Vertical truncation) As $n \to \infty$,*

$$\underline{\int}_{R^d} f(x)1_{|x|\leq n}dx \to \underline{\int}_{R^d} f(x)dx$$

(x) *(Reflection) If $f + g$ is a simple function, bounded, with finite measure support (absolutely integrable), then*

$$Simp\int_{\mathbb{R}^d} f(x) + g(x)dx = \underline{\int}_{R^d} f(x)dx + \overline{\int}_{R^d} g(x)dx$$

Lemma 11 (Markov's Inequality)**.** *Let* $f : \mathbb{R}^d \to [0, +\inf]$ *be measurable. Then for any* $0 < \lambda < \infty$*, we have*

$$m(x \in \mathbb{R}^d : f(x) > \lambda) \leq \frac{1}{\lambda} \int_{R^d} f(x)dx.$$

Proof. We have the pointwise inequality

$$\lambda 1_{\{x \in \mathbb{R}^d : f(x) \geq \lambda\}} \leq f(x).$$

From the definition of lower Lebesgue integral,

$$\lambda m(\{x \in \mathbb{R}^d : f(x) \geq \lambda\}) \leq \int_{R^d} f(x)dx.$$

□

Markov's inequality also implies that if $\int_{R^d} f(x)dx < \infty$, then f is finite almost everywhere. Also, $\int_{R^d} f(x)dx = 0$ if and only if f is zero almost everywhere.

Littlewood's three principles

(i) Every (measurable) set is nearly a finite sum of intervals;

(ii) Every (absolutely integrable) function is nearly continuous;

(iii) Every (pointwise) convergent subsequence of functions is nearly uniformly convergent.

3 Abstract measure spaces

Atomic algebra. Let X be partitioned into disjoint sets,$X = \bigcup_{\alpha \in I} A_\alpha$. We refer to each of the A_α as *atoms*. This partition generates a Boolean algebra $\mathcal{A}((A_\alpha)_{\alpha \in I})$, defined as the collection of all sets E of form $E = \bigcup_{\alpha \in J} A_\alpha$ for some $J \subset I$. We refer to this Boolean algebra as the *atomic algebra* with atoms $(A_\alpha)_{\alpha \in I}$ The trivial algebra is given by the partition $X = X$, and the discrete algebra by $X = \bigcup_{x \in X} \{x\}$. Finer or coarser partitions correspond with finer and coarser algebras respectively. Some atoms can be empty, but it makes no difference to the final atomic algebra.

Every finite Boolean algebra is an atomic Algebra, and has cardinality 2^n for some $n \in \mathbb{N}$.

Theorem 1 (Egorov's theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a finite measure space ($\mu(X) < \infty$), and let $f_n : X \to C$ be a sequence of measurable functions that converge pointwise almost everywhere to a limit $f : X \to C$. For every $\epsilon > 0$, there exists a measurable set E such that $\mu(E) \leq \epsilon$ and f_n converges uniformly to f outside of E.*

Definition 14. *An (unsigned)* ***simple function*** *$f : X \to [0, +\infty]$ on a measurable space $(X, \mathcal{B})$ is a measurable function that takes on finitely many values $a_1, ..., a_k$. We define the simple integral by*

$$Simp \int_X f d\mu := \sum_{j=1}^{k} a_j \mu(f^{-1}(\{a_j\})).$$

Theorem 2 (Monotone convergence theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $0 \leq f_1 \leq f_2 \leq ...$ be a monotone non-decreasing sequence of unsigned measurable functions on X. Then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f(x) d\mu.$$

Note that when f_n are each indicator functions, this theorem is the upwards monotone convergence property.

Theorem 3. *(Fatou's lemma) Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $f_1, f_2, ... : X \to [0, +\infty]$ be a sequence of unsigned measurable functions. Then*

$$\int_X \lim_{n\to\infty} \inf f_n d\mu \leq \lim_{n\to\infty} \inf \int_X f_n d\mu.$$

Theorem 4 (Dominated convergence theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, let $f_1, f_2, ... : X \to [0, +\infty]$ be a sequence of measurable functions that converge pointwise μ-almost everywhere to a measurable limit $f : X \to C$. Suppose there is an unsigned absolutely integrable function $G : X \to [0, +\infty]$ such that $|f_n|$ are pointwise μ-almost everywhere bounded by G for each n. Then*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$