Lebesgue Measure Revision Questions
Math 105 Real Analysis 2
Following Pugh Chp6, Tao Chp 8 and Lecture Notes 1-15

**Some of these questions may seem unnecessary, too easy and 'stupid' but this is just how I learn and make sure my basics are right.*

Lecture 1 - Introducing Outer Measure
Reading: Pugh 6.1, Tao 7.1-7.2

Pugh 6.1

1. What is outer measure used for?
2. Define the Lebesgue outer measure
3. What is the cover of an arbitrary set X?
4. Is the covering of A (as defined in 2) countable? What does it mean to be countable?
5. Define its total length.
6. Define the outer measure of A in words
7. If every series $\Sigma_k |I_k|$ diverges then m*A = ?
8. What lab equipment would you compare outer measure to?
9. [Theorem] *Axioms of Outer Measure*
    a. The outer measure of an empty set is …., … .
    b. If $A \subset B$ then …
    c. If $A = \cup_{n=1}^{\infty} A_n$ then m*A…
        i. Why is b true?
10. The area of rectangle $R = (a, b)\ x\ (c, d)$ is $|R| =$? what is the (planar) outer measure of $A \subset R^2$?
11. An open box $B \subset R^n$ is ……, B = ?
12. The n-dimensional volume |B|=?
    a. Hence, what is the n-dimensional outer measure of $A \subset R^n$?
13. If $Z \subset R^n$ has outer measure zero then it is a …. Set
14. [Proposition]
    a. Every subset of a zero set is ..
    b. The countable union of zero sets is ….
    c. Each plane $P_i(a) = \{(x_1, ..., x_n) \in R^n : x_i = a\}$ is a … set in R^n
15. [Theorem] The linear outer measure of $[a, b]$ is …; the planar outer measure of $[a, b]x[c, d]$ is …. ; the n-dimensional outer measure of a closed box is ….
16. [Corollary] The formulas $m * I =$..... , $m * R =$.... $and\ m * B =$.... hold also for intervals, rectangles and boxes that are … or ….. In particular $m * I = |I|,\ m * R\ =\ |R|,\ and\ m * B = |B|$ for x, y, z

Tao 7.1

(From intro for revision)

1. Is every piecewise continuous function Riemann integrable?
2. Is every piecewise constant function Riemann integrable?
3. How does one compute the length/area/volume of Ω?
4. What do we refer the measure of Ω as?
5. What is the geometrical meaning of a set having outer measure 0?
6. What is an example of set which has an infinite outer measure?
7. $M(A \cup B) = ?$
8. $m(A) \leq m(B)$ if…
9. $m(x + A) = ?\ (and\ in\ words)$
10. Are all open and closed sets measurable?

7.1 The Goal: Lebesgue Measure

11. For every such measurable set $\Omega \subset R^n$, the Lebesgue measure m(Ω) is a certain number in what [,]?
12. A measurable set obeys the following properties:
    a. State the Borel property: every open set in R^n is … as is every …
    b. Complementarity
    c. Boolean Algebra Property
    d. σ-algebra property
13. To every measurable set Ω, we associate the Lebesgue measure m(Ω) of Ω, which will obey the following properties (9)

7.2 Outer Measure

1. Define an open box B in R^n
2. Define vol(B)
3. In 1-dimension boxes are the same as …
4. If $b_i = a_i$ then the box is … and has volume …
5. What is $vol_1(B)$, $vol_2(B)$ or $vol_n(B)$
6. Def covering by boxes.
7. If Ω is a set ,we define the outer measure m*Ω of Ω to be the quantity..
8. List the 6 properties followed buy outer measure

Lecture 1 - Lebesgue Measure and Integral

1. $\int_\Omega f\, dx$, $\Omega \subset R^n \quad dx = dx_1 \ldots dx_n$
    a. What Ω do we allow?
    b. What f do we allow?
2. The lebesgue measure m(Ω) is the measure of that … of that set..
3. Define properties (monotone, additivity and translation invariance)

4. Properties of measurable subsets: let M_n denote the set of measurable subsets in R^n
   a. If $U \subset R^n$ is open then …
   b. If $U \in M_N$, then U^c…
   c. If $U, V \in M_n$, then …. And …. Are measurable
   d. We want M to be a $\sigma - algebra$ i.e…..
5. How to probe that Z has zero outer measure and Q.
6. Why does m*(box) = vol(box)

<u>Lecture 2 - Properties of outer measure; measurable set</u>

<u>Pugh 6.2, Tao 7.2, 7.4</u>

(Tao 7.2 already completed above)

Pugh 6.2 Measurability

1. What does it mean for two intervals to be disjoint?
2. If A and B are subsets of disjoint intervals in R then $m * (A \sqcup B) = ?$
3. Measurability is the … nonmeasurability is the ..
4. Are open sets, closed sets, their unions, differences ect measurable?
5. A set $E \subset R$ is (Lebesgue) measurable if the division …. Of R is so "clean" that for each "test set" $X \subset R$ we have m*X =
6. How do we denote the collection of all lebesgue measurable subsets of R^n
7. When do we drop the * for m*E?
8. Let M be any set.
   a. The collection of all subsets of M is denoted as ..
   b. An abstract outer measure on M is a function ω: …….that satisfies the 3 axioms of measure
   c. A set $E \subset M$ is measurable with respect to ω if E|..... we have ….
9. [Theorem] The collection M of measurable sets with respect to any outer measure on any set M is a Y And the outer measure restricted to this Y is ….. All zero sets are measurable, [do they have an effect on measurability?]
10. A $\sigma - algebra$ is a collection of sets that includes x, is closed under y and is closed under z.
    a. What does countable additivity of ω mean?
11. [Theorem] Measure Continuity Theorem. If $\{E_k\}$ and $\{F_k\}$ are sequences of measurable sets then define
    a. Upward measure continuity
    b. Downward measure continuity

Tao 7.4 Measurable Sets

1. Definition of Lebesgue measurability.
   a. We say E is measurable iff we have the identity..
   b. If E is measurable, we define the Lebesgue measure of E to be …
   c. If E is not measurable we leave m(E) …

2. Why would we subscript $m(E)$ as $m_n(E)$?
3. Are half-spaces measurable?
4. Give the 6 properties of measurable sets
    a. Compliment
    b. Translation invariance
    c. intersection/union
    d. Boolean algebra property
    e. open/closed box
    f. Zero set
5. Lemma for Finite additivity
6. Are there non-measurable sets?
7. If $A \subseteq B$ are two measurable sets
    a. Is B\A measurable?
    b. m(B\A) = ?
8. Define countable additivity [Lemma]
9. [Lemma] $\sigma - algebra$ property
10. [Lemma] every open set can be written as a countable or finite union of …
11. [Lemma] (Borel property) Every open set, and every closed set is….

Lecture 2 - Measurability
1. Definition of outer measure of any subset in R
2. Lemma 7.2.5 , proof of properties
3. Compact set in R^n $<=>$ ..
4. [Corollary] outer measure of any box = …

Lecture 3 - Tao Lemma 7.4.2-7.4.5

Tao 7.4 (Already done above)

Tao Lemmas
1. [Lemma 7.4.2] (Half spaces are measurable).
   $The\ half - space\{(x_1, ..., x_n) \in R^n: x_n > 0\}\ is\ measurable$
2. [Lemma 7.4.3]
   $A\ similar\ argument\ will\ also\ show\ that\ any\ half\ space\ of\ the\ form\ \{(x_1, ..., x_n) \in R^n: x_j > 0\}$
   $or\ \{(x_1,... x_n) \in R^n: x_j < 0\}\ for\ some\ 1 \leq j \leq n\ is\ measurable.$
3. [Lemma 7.4.4] (Properties of Measurable Sets)
    a. $If\ E\ is\ measurable,\ then\ R^n \backslash E\ is\ also\ measurable.$
    b. $(Translation\ invariance)\ if\ E\ is\ measurable,\ and\ x \in R^n,\ then\ x + E\ is\ also\ measurable,$ $and\ m(x\ E) = m(E)$
    c. $if\ E_1\ and\ E_2 are\ measurable,\ then\ E_1 \cap E_2\ and\ E_1 \cup E_2\ are\ measurable$
    d. $(Boolean\ algebra\ property)\ If\ E_{1,} E_2,..., E_N\ are\ measurable,\ then\ \cup^N_{j=1} E_j\ and\ \cap^N_{j=1} E_j\ are$ $measurable$

e. $every\ open\ box,\ and\ every\ closed\ box\ is\ measurable$
f. $Any\ set\ E\ of\ outer\ measure\ zero\ (i.e.,\ m*(E) = 0)\ is\ measurable.$

4. [Lemma 7.4.5] (Finite additivity)
   $If\ (E_j)_{j\in J}\ are\ a\ finite\ collection\ of\ disjoint\ measurable\ sets\ and\ any\ set\ A\ (not\ necessarily\ measurable)$
   $we\ have$

$$m*(A \cap \bigcup_{j\in J} E_j) = \sum_{j\in J} m*(A \cap E_j)$$

Lecture 4 - Tao Lemma 7.4.6-7.4.11

Tao 7.4 Done already

1. [Remark 7.4.6]
   $Lemma\ 7.4.5\ (Finite\ additivity)\ and\ Proposition\ 7.3.3\ when\ combined,\ imply\ that\ there\ exists$
   $non-measurable\ sets.$
   $Proposition\ 7.3.3\ (Failure\ of\ finite\ additivity).\ There\ exists\ a\ finite\ collection\ (A_j)_{j\in J}\ of\ disjoint$

   $subsets\ of\ R,\ such\ that \quad m*(\bigcup_{j\in J} A_j) \neq \sum_{j\in J} m*(A_j)$
2. [Corollary 7.4.7]
   $If\ A \subseteq B\ are\ two\ measurable\ sets,\ then\ B\backslash A\ is\ also\ measurable,\ and$
   $$m(B\backslash A) = m(B) - m(A)$$
3. [Lemma 7.4.8] (Countable additivity)

   $If\ (E_j)_{j\in J}\ are\ a\ countable\ collection\ of\ disjoint\ measurable\ sets,\ then\ \bigcup_{j\in J} E_j\ is\ measurable,$

   $and\ m(\bigcup_{j\in J} E_j) = \sum_{j\in J} m(E_j)$
4. [Lemma 7.4.9] $\sigma - algebra\ property$
   $If\ (\Omega_j)_{j\in J}\ are\ any\ countable\ collection\ of\ measurable\ sets\ (so\ J\ is\ countable),\ then\ the\ union$

   $\bigcup_{j\in J} \Omega_j\ and\ the\ intersection\ \bigcap_{j\in J} \Omega_j\ are\ also\ measurable.$
5. [Lemma 7.4.10]
   $Every\ open\ set\ can\ be\ written\ as\ a\ countable\ or\ finite\ union\ of\ open\ boxes.$
6. [Lemma 7.4.11] (Borel Property)
   $Every\ open\ set,\ and\ every\ closed\ set,\ is\ Lebesgue\ measurable.$

Lecture 5 - Measurable Function, Regularity

Pugh 6.4, Tao 7.5

Pugh 6.4 Regularity

1. [Theorem] Open sets and closed sets in R^n are ….
2. [Proposition]
   $The\ half-spaces\ [a, \infty) \times R^{n-1}\ and\ (a, \infty) \times R^{n-1}\ are\ .....\ in\ R^n.\ So\ are\ all\ open\ ...$
3. Do zero sets have any effect on outer measure?

4. Is $\sigma - algebra$ closed with respect to
    a. countable unions?
    b. Complements?
5. [Corollary] The Lebesgue measure of a closed or partially closed box is the volume of it's …. The boundary of a box is a …
6. What is the countable intersection of open sets called?
7. What is the countable union of closed sets called?
8. What is the complement of a (6) set? Is it true conversely?
9. A homeomorphism sets G-sets to G-sets and F-sets to F-sets. What is a homeomorphism?
10. Does σ-algebra contain G and F sets? Why?
11. [Theorem]
    $Lebesgue\ measure\ is\ regular\ in\ the\ sense\ that\ each\ measurable\ set\ E\ can\ be$ ...., $s.t.\ G\backslash F\ is\ a$ .... $set.\ Conversely,\ if\ there\ is\ such\ an$ .... $then\ E\ is\ measurable.$
12. [Corollary]
    $A\ bounded\ set\ E \subset R^n\ is\ measureable\ iff\ it\ has\ a$ .... $s.t\ F\ is$ .. , $G\ is$ ..., $and\ mF =$...
13. [Corollary] $Modulo\ zero\ sets,\ Lebesgue\ measurable\ sets\ are$ ... and ...
14. [Corollary] $A\ Lipeomorphism\ h: R^n -> R^n$ is a ….
15. Define a lipeomorphism
16. Define a mesomorphism

Affine Motions

17. What is an affine motion of R^n?
18. Does translation affect Lebesgue measure?
19. [Lemma] $Every\ open\ set\ is\ a\ countable\ disjoint\ union\ of$... $plus\ a$ ....
20. [Corollary] $Rigid\ motions\ of\ R^n\ preserve$.... $They\ are\ me$...

Inner Measure, Hulls, and Kernals

21. Consider any bounded $A \subset R^n$, measurable or not. m*A is the infimum of the measure of ….. That contain A
22. The infimum is achieved by a … what do we call it?
23. Define a Hull.
24. The inner measure of A is the the supremum of the measure of ….The supremum is achieved by …… what do we call it?
25. $m_*A = ?$
26. How does m_* measure A?
27. Is it greater than or less than the outer measure m* ?
28. Is m_* monotone? If yes what does this imply
29. Definite the measure theoretic boundary of set A
30. [Theorem] $If\ A \subset B \subset R^n\ and\ B\ is\ a\ box\ then\ A\ is\ measurable\ iff$.....
31. [Lemma] $If\ A\ is\ contained\ in\ a\ box\ B\ then\ mB = ?$

Tao 7.5 - Measurable Functions

1. Define measurable functions
2. Are continuous functions measurable? [Lemma]
3. [Lemma 7.5.3]
   $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n,\ and\ let\ f:\ \Omega\ -> \ R^m\ be\ a\ function.\ Then\ f\ is\ measurable\ iff...$
   $for\ every\ open\ box\ B$

4. [Lemma 7.5.4]
   $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n,\ and\ let\ f: \Omega\ -> \ R^m\ be\ a\ function.\ Suppose\ that\ f\ =\ (f_1,..., f_m)$
   $where\ f_j: \Omega\ -> \ R\ is\ the\ j^{th}\ co\ -\ ordinate\ of\ f.\ Then\ f\ is\ measurable\ iff...$
5. Is the composition of two measurable functions measurable?
6. [Lemma 7.5.5]
   $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n,\ and\ let\ W\ be\ an\ open\ subset\ of\ R^m.\ If\ f: \Omega\ -> \ W\ is\ measurable$
   $and\ g: W\ -> R^p\ is\ continuous,\ then\ is\ g\ \circ\ f: \Omega\ -> R^p\ measurable?$
7. [Lemma 7.5.6]
   $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n.\ If\ f: \Omega\ -> \ R\ is\ a\ measurable\ function,\ then\ is\ |f|,\ max(f, 0)$
   $min(f, 0)?$
8. [Corollary 7.5.7]
   $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n.\ If\ f: \Omega\ -> \ R\ and\ g: \Omega\ -> \ R\ are\ measurable\ functions\ then$
   $is\ f\ +\ g, f\ -\ g, fg, max(f, g)\ ,\ min(f, g)?\ when\ is\ f/g\ measurable?$
9. [Lemma 7.5.8]
   $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n,\ and\ let\ f: \Omega\ -> \ be\ a\ function.\ Then\ ......\ iff\ f^{-1}((a, \infty))\ is$
   $measurable\ for\ every\ ....$
10. [Lemma 7.5.9] (Measurable functions in the extended reals)
    $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n.\ A\ function\ f: \Omega\ -> \ R\ *\ \ is\ said\ to\ be\ measurable\ iff\ .....\ is$
    $measurable\ for\ every\ real\ number\ a.$
11. What is R*?
12. [Lemma 7.5.10] (Limits of measurable functions are measurable)
    $Let\ \Omega\ be\ a\ measurable\ subset\ of\ R^n.\ For\ each\ positive\ integer\ n,\ let\ f_n: \Omega\ -> \ R\ *\ \ be\ a$
    $measurable\ function.\ Then\ are\ sup_{n\geq1}f_n,\ inf_{n\geq1}f_n,\ limsup_{n\to\infty}f_n, liminf_{n\to\infty}f_n\ also\ measurable?$
    $if\ f_n\ converge\ pointwise\ to\ another\ function\ f: \Omega\ \to\ R\ *,\ is\ f\ also\ measurable?$

<u>Lecture 6 Products & Slices</u>

Pugh 6.5

1. [Theorem] Measurable Product Theorem
   m(A x B) = ?
2. [Lemma] If A and B are boxes then A x B is measurable and m(A x B) = ?
3. [Lemma] If A and B are zero sets then what is A x B?

4. [Lemma] If U and V are open:
    a. Is U x V measurable?
    b. What is m(U x V)?
5. The hull of a product is the product of ….
6. The kernel of a product is the product of …
7. What set is the slice of $E \subset R^n \times R^k$ $at\ x \in R^n$?

8. [Theorem] Zero Slice Theorem
   $If\ E \subset R^n\ x\ R^k\ is\ measurable\ then\ E\ is\ a\ zero\ set\ iff$ ....
9. [Lemma] $If\ W \subset I^n\ is\ open\ and\ X_\alpha = X_\alpha(W) = \{x: m(W_x) > \alpha\}\ then\ mW \geq$....

Lecture 6 Questions

1. If $E\ c\ R^n$ is measurable if $\exists\ a\ G_\delta -\ set\ G\ and\ F_\delta -\ set\ F$ s.t
    a. What is the relation between E,F,G?
    b. m(G/F)=?
2. What is the relation between G and F (2)
3. If G1, G2 are nulls of F, then what are G1/G2, G2/G1?
4. If G1, G2, … are $G_\delta\ sets$ then $\cap\ G_i$ is a ….
5. If F1, F2, … are $F_\sigma\ sets$ then $\cup\cup\ F_{ij}$
6. Draw a slice in a diagram
7. $If\ E\ c\ R^n,\ F\ c\ R^k$ are measurable sets then:
    a. m(E x F) = ?
    b. If m(E)=0 or m(F)=0 then what is m(E x F)=?
8. If $E\ c\ R\ has\ m(E),\ m(E\ x\ R) =?$
9. What are the 3 situations when m(E x F) = m(E)m(F)?
10. Any open set $U \subseteq R^n$ can be written as (1) and measure … set
11. Does $m(H_E\ x\ H_F \setminus K_E x K_F) = 0?$

Lecture 7 Lebesgue Integral

Pugh 6.6 - Lebesgue Integral

1. Define the undergraph of f
2. The function f is (Lebesgue) measurable if Uf is ….., and if it is then the lebesgue integral of f is ….
3. The undergraph is the …. Set of f
4. Can an infinite set have infinite measure?

    a. Hence what can $\int f =?$

5. Define a Lebesgue Integrable function
    a. Does the integral of a measurable non negative function exist even if the function is not integrable?

6. [Theorem] Monotone convergence theorem: Assume fn is a sequence of measurable functions and … as n goes to infinity. Then …
7. Define the completed undergraph
8. The completed undergraph is measurable iff … is measurable, and if it is their … are equal
9. $f_n \uparrow f$ implies Uf_n...
10. The completed undergraph = what intersection ? except for what points?
11. Does the x axis have an effect on measurability?
12. What does the measure of the completed undergraph equal
13. [Corollary] $If\ (f_n)\ is\ a\ sequence\ of\ integrable\ functions\ that\ converges\ monotonically\ downward\ to\ a\ limit\ function\ f\ almost\ everywhere\ then$ ...
14. Define the envelope sequences of the function f
15. U(fn (upper)) = U ?
16. U[closed](fn (lower)) =∩?
17. [Theorem] Dominated Convergence Theorem
18. [Corollary] The pointwise limit of measurable functions is ….
19. [Lemma] Fataou's Lemma
20. [Theorem] 8 properties about measurable functions f, g
21. Define the f-translation T
    a. Tf slides points along …
    b. $T_f \circ T_g =\dots=\dots$
22. If f is integrable then Tf preserves ……
23. [Corollary] If fk is a sequence of integrable functions then $\sum_{k=0}^{\infty} \int f_k =?$
24. If f takes both positive and negative values we define $f_+(x)$, $f_-(x)$
    a. Define $\int f\ in\ terms\ of\ f_+\ and\ f_-$

<u>Lecture 7 Slices/Lebesgue Integral</u>

1. m(E) = 0 iff ….
2. If $z = \emptyset$, is E bounded or unbounded?
3. Define undergraph U(f)
    a. We say f is measurable if U(f) is …
    b. $\int f :=$ ......
        i. Can this equal infinity?
    c. If $\int f < \infty$, what can we say about f?
4. What does a.e = "almost everywhere" mean?

a. Give an example for f(x) = 0 a.e

5. [Theorem 27]
   $let\ f_n = R \rightarrow [0, \infty)\ be\ a\ sequence\ of\ measurable\ functions\ and\ fn\nearrow f\ a.e\ as\ n \rightarrow \infty$

   $then \int f_n$ converges to what?
6. Define a completed undergraph $\hat{U}(f)$
7. [Proposition] if $f_n: R \rightarrow [0, inf]$ is a sequence of integrable functions that fn converges

   to f a.e, what does the $\int f_n$ converge to?
8. If $f_n(x)$ is a sequence of functions then what is $f^{bar}{}_n(x)$ and $f_{n, bar}(x)$ (couldn't write these properly on G.docs)
9. $U(f^{bar}{}_n)$ = ?
10. $\hat{U}(f)$=?
11. [Theorem] if $f_n: R \rightarrow [0, inf]$ is a sequence of integrable functions that fn converges to

    f a.e, $g(x) \geq f_n(x)\ a.e\ ,\ \int g\ < \infty\ then \int f_n \rightarrow \int f$

    What can we say about U(fn) and m(U(fn)) and their relation to g, U(g)

<u>Lecture 8 - Lebesgue Integral</u>
<u>Pugh 6.6 (*Done Above*)</u>

Lecture 8 - Lebesgue Integral

1. Define a Lebesgue Integral
2. State the dominated convergence theorem
3. If g is integrable => $\int g$....
4. [Corollary] The pointwise limit of measurable functions is …..
5. [Factous Lemma] $f: R \rightarrow [0, \infty)$ is measurable and fn is measurable then

   $\int lim\ inf\ f_n\ \leq$
6. [Theorem] if f,g are measurable then $\int f\ +\ g\ =?$
7. What is a mesomorphism?
8. What is mesometry?
9. Is the translation of a box measurable?

Lecture 9 - Simple Functions

Tao 8.1

1. Define simple functions
2. Definite the characteristic function
3. Give the 3 basic properties of simple functions
4. State the 3 lemmas about properties of simple functions
5. Define the lebesgue integral of a simple function
6. Let Ω be a measurable set, and let f and g be non-negative simple functions. Give the 4 properties

Lecture 9 - Simple Functions

1. Define an indicator function/ characteristic function
2. $1_E(x) = \{ 1 \ldots\ldots\ldots, 0\ldots\ldots\}$
3. Define simple functions in terms of indicators
4. $\int 1_E$ =?
5. $\int \sum c_i 1_E$ =?
6. For a non-negative measurable function
    a. Simple function $f_n: R^n \to [0, \infty)\ s.t.\ f_n \nearrow$...
    b. Define $\int f = lim$..
7. [Tao 7.5] Definition of a measurable function.
   $f: R^n \to R$ is measurable if for all open sets V c R, ….
8. $\Omega\ c\ R^n$ a measurable set, $f: \Omega \to R$:
    a. Is f measurable?
    b. Is $f^{-1}(open)$ measurable?
9. [Lemma] $f: \Omega \to R^n$ measurable $f: \Omega \to R$ continuous, then is f measurable?
10. If $f: \Omega \to R^k$ is measurable and g: $f(\Omega) \to R^L$, then is $g \circ f$ measurable?
11. [Corollary] if f is measurable $R \to R$, then is |f| measurable?
12. If f1, f2 are measurable $R \to R$ then f1 +f2 is measurable
13. [Lemma] a function $f: R^k \to R^n, f = (f_1, \ldots\ldots, f_n)$ is measurable iff ?
14. Let $f: R \to [0, \infty)\ Uf\ measurable \Leftrightarrow \forall V\ open\ in\ R^+$....... $\Leftrightarrow \forall (a, \infty)\ a \geq 0$
15. $f^{-1}((-\infty, a]) = [..]^c$ is measurable
16. $a < b\ then\ f^{-1}((a, b]) = f^{-1}((..)/(...)) = f^{-1}(....)/f^{-1}(.....)$
17. If $E\ c\ R^2$ is measurable, is $\pi i(E)$ measurable?
18. If $f: R \to R^2$ continuous function $E\ c\ R^2$ measurable, is $f^{-1}(E)\ c\ R$ measurable?
    a. Give an example

19. Is the pre image of measurable sets always measurable for measurable functions?
20. What is a cantour set?
21. What is a homomorphism?
22. If f is a simple function, then $\exists\, E_1,\ \ldots\ldots,\ E_n$ disjoint meas. Subsets of R

    $c_1, \ldots, c_n \in R\ s.t.\ f(x) = \sum \ldots$

23. [Proposition] the set of simple functions form a vector space i.e.
    $\forall\, c \in R\ \ \forall\, f$ *are simple functions*, (i) *are cf simple*? (ii) *if g is simple, are* $f + g$ *simple*?

24. Let f be a simple function where λ is the height. What is $\int f$ =?

25. Let f be measurable, then there exists fn sequence of nonmeasurable simple functions of *bounded support* st fn converges to what and how?
    a. What does bounded support mean?
26. f is a pointwise limit of a sequence of simple functions ⇔?

Lecture 10 Lebesgue Integral

Tao 8.2 Integration of non-negative measurable functions

1. Define Majorization
2. Define Lebesgue integral for non-negative functions
3. Let Ω be a measurable set, and let f:Ω $\rightarrow [0, \infty]$ and g:Ω $\rightarrow [0, \infty]$ be non-negative measurable functions. State the 5 properties (iff f(x)=0, c>0, f<g, f=h,:Ω '
4. [Theorem] Lebesgue Monotone Convergence Theorem
5. [Lemma] Interchange of addition and integration.
6. [Lemma] Fataou's Lemma
7. [Lemma] … then f is finite almost everywhere
8. [Lemma] Borel-Cantelli
9.

Lecture 10

1. If f and g are simple functions $\int f + g = ?$

2. Let $f \geq 0$ be measurable $f\colon \Omega \rightarrow [0, \infty)$. $\int f = sup\{\int s \mid s \ldots\ldots\}$

3. [Proposition] if $f, g\colon \Omega \rightarrow [0, \infty]$

    a. $\int f \geq 0$ *and* $\int f = 0$ iff ….

    b. For all c>0 $\int cf = ?$

4. $f \le g \Rightarrow \int f \ ? \ \int g$

5. If f=g a.e then $\int f$? $\int g$

6. [Theorem] Given measurable functions $f: \Omega \to [0, \infty]$ $f_n: \Omega \to [0, \infty]$

   $0 \le f_1(x) \le f_2(x) \le ...$

   Then $\int sup\, f_n(x) = ?$

7. Show this ^^

8. If $f: \Omega \to [0, \infty]$ is measurable $\int f < \infty$ then f(x) is …. A.e.

9. [Corollary] Borel - Cantelli

Lecture 11 - Dominated Convergence theorem, Fubini Theorem

Tao 8.5, Pugh 6.7

Pugh 6.7 Italian Measure Theory

1. Definte x-slice and y-slice
2. State Cavalieri's Principle. (measureable slices)
3. [Corollary] The y-slices of an undergraph decrease monotonically as y ….., and the following formulas hold:
4. What does it mean to be preimage measurable?
5. [Theorem] State the fubini-tonelli theorem
6. [Corollary] Does the order of integration count for measurable functions?

Tao 8.5 Fubini's Theorem

1. Once we know how to integrate on R^2 , we can integrate on …. Of R^2. why?
2. State fubini's theorem

Lecture 11

1. Given $f: R^n -> R$, define:
   a. E_+, E_-
   b. f_+, f_- *and hence f*
2. Define an absolutely integrable function
3. State the dominated convergence theorem (hint:lim…)
4. State Fatou's Lemma

L12 Vitali Covering, Upper and Lower Lebesgue Integrals

Pugh 6.8 Vitali coverungs and density points

1. Every open covering of a closed and bounded subset of Euclidean space reduces to a ……
2. Define a vitali covering
3. State the vitali covering lemma
   a. What are the three characteristics
4. State the vitali covering lemma for cubes
5. Define the density of E, $E \subset R^n$
6. Define the density points of E
7. Define the concentration of E in Q
8. What is balanced density?
9. State the Lebesgue Density Theorem
10.

Lecture 12

1. Define the upper and lower lebesgue integrals of any function f
   a. Give the 2 properties of these
2. [Lemma] if the upper and lower lebesgue integrals are equal, then what exists? And where?
3. [Theorem] Fubini Theorem
4. Define a vitali covering
   a. Are the elements always open?
5. [Theorem] Vitali Covering Lemma

L13 Vitali Covering, Lebesuge Density Theorem

Pugh 6.8

Lecture 13

1. *Proving the Vitali Covering Lemma*

L14 Lebesgue Mean Value Theorem, Density Theorem

Pugh 6.9 - Calculus à la Lebesgue

1. Define the average of a locally integrable function over a measurable set A
2. How do you denote the concentration of f on A?
3. What do we mean by locally integrable?
4. State the average value theorem
5. Definite absolutely continuous
6. Every absolutely continuous function is …. Continuous. If (Ii) is a sequence of disjoint intervals then the following are equivalent for a continuous function G:
7. If G is absolutely continuous and Z is a zero set, what does GZ=?

8. If G is absolutely continuous then it is ….continuous in the sense that given $\epsilon > 0$, there exists a $\delta > 0$ such that mE<δ => ?
9. [Theorem] Let f be a Lebesgue integrable and let F be its indefinite integral F(x) = …. Give 3 properties:
    a. For almost every x the derivative F'(x) = ?
    b. F is … continuous
    c. If G is an absolutely continuous function and G'(x)=f(x) for almost every x then G differs from ….
10. State Lebesgue's Antiderivative Theorem

Lecture 14
1. What does the density theorem say? (normal words)
2. Recall the formula for the approximate density
3. What are the properties of the approximate density δ?
    a. (range)
    b. (what happens if it exists)
4. [Theorem] If E is measurable, then almost all … i.e …….
5. [Lemma] if ….. Then dp(E1) = dp(E2)
6. [Theorem] let f be locally integrable then for a.e p…… exists and it equals …
7. If $g: R^n -> [0, \infty)$ $integrable\ for$ $\forall\ \alpha > 0$ then …. n*(..... <....

<u>Lecture 15 - Absolute Continuous Function, Lebesgue Main Theorem</u>

Pugh 6.9 (Done)

Lecture 15
1. What is the Lebesgue density theorem?
2. Define an absolutely continuous function f
3. Where does an absolutely continuous function send a null set to?