Exercise 7.2.1

(v) Prove that the empty set $\emptyset$ has measure $m^*(\emptyset) = 0$:

all open sets cover $\emptyset$, including those with zero volume, so

$$\inf\left\{ \sum_i |I_i| : (I_i)_{i\in J} \text{ covers } \emptyset \right\} = 0$$ □

(vi) Prove that $0 \leq m^*(\Omega) \leq +\infty$ for all measurable set Ω:

Since $Vol(\cdot)$ is a nonnegative function, the sum of volumes can never be negative □

(vii) Proof that if $A \subseteq B \subseteq \mathbb{R}$, then $m^*(A) \leq m^*(B)$:

Since A is a subset of B, B covers A.

Thus we find that

$$m^*(A) = \inf\left\{ \sum_j Vol(I_j^A) : (I_j^A)_{j\in J} \text{ covers } A \right\}$$

$$\leq \inf\left\{ \sum_j Vol(I_j^B) : (I_j^B)_{j\in J} \text{ covers } B \right\}$$

$$= m^*(B)$$ □

(viii) Prove: If $(A_j)_{j\in J}$ are a finite collection of subsets of $\mathbb{R}^n$, then

$$m^*\left(\bigcup_{j\in J} A_j \right) \leq \sum_j m^*(A_j)$$

We know that the collection is finite, and that for all A_j exists $(I_{kj})_{k\in K}$ such that
$m^*(A_j) = \sum_k Vol(I_{kj})$
for an $\varepsilon > 0$, it holds that for all $j \in J$:

$$\sum_k Vol(I_{kj})_{k\in K} \leq m^*(A_j) + \frac{\varepsilon}{|J|}$$

Since A_j is covered by $(I_{kj})_{k\in K}$ and $A = \bigcup_{j\in J} A_j$, $(I_{kj})_{k\in K, j\in J}$ covers A and

$$m^*(A) \leq \sum_j \sum_k Vol(I_{kj}) \leq \sum_j \left(m^*(A_j) + \frac{\varepsilon}{|J|}\right) = \sum_j m^*(A_j) + \varepsilon$$

And as $\varepsilon \to 0$ we find that

$$m^*\left(\bigcup_{j\in J} A_j\right) = m^*(A) \leq \sum_j m^*(A_j)$$

(x) Using the same argument as previously we now have a countable set $(A_j)_{j\in J}$. For every A_j exists $(I_{kj})_{k\in K}$ such that for every $\varepsilon > 0$

$$\sum_k Vol(I_{kj}) < m^*(A_j) + \frac{\varepsilon}{2^j}$$

Again, $(I_{kj})_{k\in K, j\in J}$ covers A and

$$m^*(A) \leq \sum_j \sum_k Vol(I_{kj}) < \sum_j \left(m^*(A_j) + \frac{\varepsilon}{2^j}\right) = \sum_j m^*(A_j) + \varepsilon$$

and as J is countable, $\sum_j \frac{\varepsilon}{2^j} = \varepsilon$. As $\varepsilon \to 0$, we find
$$m^*\left(\bigcup_{j\in J} A_j\right) = m^*(A) \leq \sum_j m^*(A_j)$$

(xiii)

Let $(I_j)_{j\in J}$ be a collection that covers Ω.

If we translate Ω by x to $\Omega + x$ we find that

$$m^*(\Omega + x) \le \sum_j \mathrm{Vol}(I_j + x)_{j\in J} = \sum_j \mathrm{Vol}(I_j)_{j\in J} + \sum_j \mathrm{Vol}(x)$$
$$= \sum_j \mathrm{Vol}(I_j)_{j\in J} = m^*(\Omega)$$

If we let $T = \Omega + x$ and let $(I_j)_{j\in J}$ cover T.

Then $(I_j - x)_{j\in J}$ covers Ω and

$$m^*(\Omega) \le \sum_j \mathrm{Vol}(I_j - x)_{j\in J} = \sum_j \mathrm{Vol}(I_j)_{j\in J} = m^*(\Omega + x)$$

And by this

$$m^*(\Omega) = m^*(\Omega + x)$$

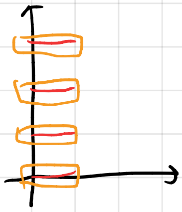
Exercise 7.2.2

are boxes ↓

$0 = \inf \sum$ [box] $= \sum_j ($[box] $\times$ [interval]$)_j$

$= \sum \mathrm{Vol}($[box]$)\,\mathrm{Vol}($[interval]$)$

$(0,1) \times (0,1) = ((0,0), (1,1)) \in \mathbb{R}^2$

$(0,1) \times \mathbb{N} = ((0,1), (0,2), (0,3), \dots) \simeq$ essentially 

[box] = [red box]

↑ thought proces of the proof: We can cover $A \times B$ with $A_j \times B_k$

⁎ Let $(A_j)_{j\in J}$ be open boxes covering A and

⁎ let $(B_k)_{k\in K}$ be open boxes covering B.

Then $\sum_j \mathrm{Vol}(A_j) < m_n^*(A) + \varepsilon$ and $\sum_k \mathrm{Vol}(B_k) < m_m^*(B) + \varepsilon$ for $\varepsilon > 0$

Because of ⁎ and ⁎ we know that

$$\bigcup_{i,j} A_j \times B_j \text{ covers } A \times B.$$

Thus by lemma 7.2.5 (x)

$$m^*_{n+m}(A\times B)=m^*_{n+m}\left(\bigcup_{j,k} A_j\times B_k\right)\le\sum_{j,k} m^*_{m+n}(A_j\times B_k)$$

$$\overset{*}{=}\sum_{j,k} m^*_n(A_j)\,m^*_m(B_k)=\sum_j m^*_n(A_j)\sum_k m^*_m(B_k)$$

$$<(m^*_n(A)+\varepsilon)(m^*_m(B)+\varepsilon)$$

* because we use open boxes, we just multiply areas of boxes

and as we let $\varepsilon\to 0$ we find that

$$m^*_{n+m}(A\times B)\le m^*_n(A)\,m^*_m(B)$$

Exercise 9.2.3

a) ex for case of $A_1\subseteq A_2$. By monotonicity,

$m^*(A)\le m^*(B)$ We need to show that

$m^*(A\cup B)=m^*(B)$.

As $A\subseteq B$, $A\cup B=B$ and the result is trivial

Prove that for $A_1\subseteq A_2\subseteq\ldots$ we have $m^*(\bigcup_j A_j)=\lim_{j\to\infty} m^*(A_j)$

PROOF by induction on j

$j=2$: $A_1\subseteq A_2$. As A_2 covers A_1 and $A_1\subseteq A_2\Leftrightarrow A_1\cup A_2=A_2$

we have that

$$m^*(A_1\cup A_2)=m^*(A_2)$$

Induction step - $k=n+1$ * from 2

Assume that *up until n, $m^*(\bigcup_j^n A_j)=m^*(A_n)$,

then by using our induction hypothesis we find that since $A_n\subseteq A_k\Leftrightarrow A_n\cup A_k=A_k$, so

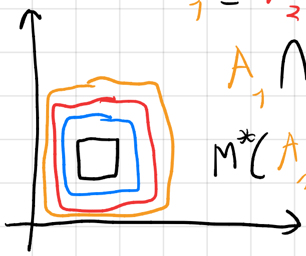
$$m^*\left(\bigcup_j^k A_j\right)=m^*(A_n\cup A_k)=m^*(A_k)$$ □

b) Prove that for $A_1 \supseteq A_2 \supseteq \dots$

$$m^*\left(\bigcap_j^{\infty} A_j\right) = \lim_{j \to \infty} m^*(A_j)$$

sketch

$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots \supseteq A_n$



$A_1 \cap A_2 = A_2$

$M^*(A_1 \cup A_2) = m^*(A_2)$

Same as before

PROOF by induction on j

$j = 2$: $A_1 \supseteq A_2 \Leftrightarrow A_1 \cap A_2 = A_2$

which means that

$$m^*\left(\bigcup_j^2 A_j\right) = \lim_{j \to 2} m^*(A_j) = m^*(A_2)$$

Induction step: $k = n+1$

Assume that from 2 to n it holds that

$$m^*\left(\bigcup_j^n A_j\right) = \lim_{j \to n} m^*(A_j)$$

then by using our induction hypothesis on A_k, since $A_n \supseteq A_{n+1} = A_k$ we find that

$$A_n \cap A_k = \bigcap_j^n A_j \cap A_k = \bigcap_j^k A_j = A_k$$

and therefore

$$m^*\left(\bigcap_j^k A_j\right) = \lim_{j \to k} m^*(A_j)$$

□

Exercise 7.2.4

Prove that for any $q > 1$, that the open box

$(0, 1/q)^n = \{(x_1 \cdots x_n) \in R^n : 0 < x_j < 1/q \text{ for } 0 \leq j \leq n\}$

and the closed box

$[0, 1/q]^n = \{(x_1 \cdots x_n) \in R^n : 0 \leq x_j \leq 1/q \text{ for } 0 \leq j \leq n\}$

both measure q^{-n}

PF: open box) Imagine an open box $[0,1]^n$ with measure 1. This box can be covered by q^n boxes of size $(0, 1/q)^n$ such that each box is translated by n/q, $n \in \mathbb{Z}_0^+$ in each direction. Since each box is disjoint and together are a subset of $[0,1]^n$ by monotonicity we get

$m^*([0,1]^n) = 1 \geq q^n m^*((0, \frac{1}{q})^n) \Leftrightarrow m^*((0, 1/q)^n) \leq q^{-n}$. *

Now if we look at q^n translates of the open box $[0, 1/q]^n$, since $\bigcup_n ([0, \frac{1}{q}] + \frac{n}{q}) = [0,1]^n$, by sub additivity we get

$m^*([0,1]^n) = 1 \leq q^n m^*([0, \frac{1}{q}]) \Leftrightarrow m^*([0, 1/q]) \geq q^{-n}$.

Exercise 7.4.1

Show that if A is an open interval in $\mathbb{R}$, then $(0,\infty)$ is measurable, i.e. $m^*(A) = m^*(A \cap (0,\infty)) + m^*(A/(0,\infty))$

we note that $A/(0,\infty) = A \cap (-\infty, 0]$. As in the lectures, Lets define that $A_+ = A \cap (0,\infty)$ and $A_- = A \cap (-\infty, 0]$

Since $A_+ \sqcup A_-$, by sub additivity

$$m^*(A) \leq m^*(A_+) + m^*(A_-) = m^*(A \cap (0,\infty)) + m^*(A/(0,\infty))$$

now to show $\geq$ consider a collection $(B_j)_{j\in J}$ that covers A such that

$$* \sum_j Vol(B_j) \leq m^*(A) + \frac{\varepsilon}{2},$$

Now let $B_j^+ = B_j \cap (0,\infty)$ and $B_j^- = B_j \cap (\infty, 0]$

then $B_j = B_j^+ \cup B_j^-$ and using another ε

$$Vol(B_j) + \frac{\varepsilon}{2^{j+1}} \geq Vol(B_j^+) + Vol(B_j^-) \geq Vol(B_j)$$

Now since $\bigcup_j B_j^+ \supseteq A_j^+$ and $\bigcup_j B_j^- \supseteq A_j^-$ we find

$$m^*(A^+) + m^*(A^-) \leq \sum_j Vol(B_j^+) + \sum_j Vol(B_j^-) \leq \sum_j Vol(B_j) + \frac{\varepsilon}{2^{j+1}}$$

$$\leq \sum_j Vol(B_j) + \frac{\varepsilon}{2} \overset{*}{\leq} m^*(A) + 2\frac{\varepsilon}{2} = m^*(A) + \varepsilon$$

and by letting $\varepsilon \to 0$ we find that

$$m^*(A) \geq m^*(A \cap (0,\infty)) + m^*(A/(0,\infty))$$

$\square$

Exercise 7.4.2

Prove that if A is an open box in R^n, and E is the half plane $E=\{(x_1\cdots x_n)\in R^n : x_n>0$, then E is measurable.

Pf: Let $A_+ = \{(x_1 \cdots x_n)\in R^n : x_n > 0\}$,

$A_- = \{(x_1 \cdots x_n)\in R^n : x_n \le 0\}$

we then want to prove that

$$m^*(A) = m^*(A\cap A_+) + m^*(A/A_+) = m^*(A_+) + m^*(A_-)$$

we know that $m^*(A) = Vol(A)$, $m^*(A_+) = Vol(A_+)$, and $m^*(A_-) = Vol(A_-)$ by definition of an open box.

We first show that $m^*(A) \ge m^*(A_+) + m^*(A_-)$:

for any $\varepsilon_1 > 0$, find a $\{B_j\}$ that covers A such that

$$m^*(A) + \varepsilon_1 \ge \sum_i Vol(B_i) = \sum_i (Vol(B_i^+) + Vol(B_i^-))$$
$$= \sum_i Vol(B_i^+) + \sum_i Vol(B_i^-) \; *$$

Where $\{B_i^+\}$ then covers A_+ * and $\{B_i^-\}$ covers A_- * , and where

$B_i^+ = B_i \cap \{(x_1 \cdots x_n) : x_n > 0\}$

$B_i^- = B_i \cap \{(x_1 \cdots x_n) : x_n \le 0\}$

take another $\varepsilon_2 > 0$ and by * and *

$m^*(A_+) \le \sum_i m^*(B_i^+) = \sum_i Vol(B_i^+)$

$m^*(A_-) \le \sum_i m^*(B_i^-) = \sum_i Vol(B_i^-)$

$\Leftrightarrow$ $m^*(A_+) + m(A_-) \le m^*(A) + \varepsilon_2$ and when $\varepsilon_1 . \varepsilon_2 \to 0$

$m^*(A) = m^*(A_+) + m^*(A_-)$ $\square$

Exercise 7.4.3

Prove that the half space $\{(x_1 \cdots x_n) \in R^n : x_n > 0\}$ is measurable.

Pf. This follows from exercise 7.4.2, since for an open box A,

$$m^*(A) = m^*(A_+) + m^*(A_-)$$

with A_+ being the half space and $A_- = A / A_+$ □

Exercise 7.4.4

Prove Lemma 7.4.4

a) if E is measurable, then E^c is measurable

Pf: We have that

$$m^*(A) = m^*(A \cap E) + m^*(A/E)$$
$$= m^*(A/E^c) + m^*(A \cap E^c)$$ □

b) If E is measurable and $x \in R^n$, then $x + E$ is measurable and $m(x+E) = m(E)$

Pf: We know that outer measure is translation invariant, i.e., $m^*(A+x) = m^*(A)$. We have that

$$m^*(A) = m^*(A \cap (x+E)) + m^*(A/(E+x))$$
$$\Leftrightarrow m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x)/E)$$

and by translation invariance of OM, it holds □

c) Prove that if E_1 and E_2 measurable then $E_1 \cap E_2$ and $E_1 \cup E_2$ measurable.

Pf for $E_1 \cap E_2$: wts $m^*(A) = m^*(A \cap E_1 \cap E_1) + m^*(A / (E_1 \cap E_2))$

we start by noticing that

$$(E_1 \cap E_2)^c = (E_1 \cap E_2^c) \cup (E_1^c \cap E_2) \cup (E_1^c \cap E_2^c)$$

we define

$$A_{++} = A \cap E_1 \cap E_2, \quad A_{+-} = A \cap E_1 \cap E_2^c,$$
$$A_{-+} = A \cap E_1^c \cap E_2, \quad A_{--} = A \cap E_1^c \cap E_2^c$$

and notice that $A_{++} \sqcup A_{-+} \sqcup A_{+-} \sqcup A_{--}$.

With these definitions we now need to show

$$m^*(A) = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})$$

Looking at $m^*(A)$ we can deconstruct it bc disjoint

$$* \; m^*(A) = m^*(A_{++}) + m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$$

and by the same argument we have

$$* \; m^*(A_{+-} \cup A_{-+} \cup A_{--}) + m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$$

By * and * we have that

$$m^*(A) = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})$$

Pf for $E_1 \cup E_2$: wts $m^*(A) = m^*(A_{++} \cup A_{+-} \cup A_{-+}) + m^*(A_{--})$

using the same argument as above, bc disjoint set

$$m^*(A) = m^*(A_{++}) + m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$$
$$= m^*(A_{++} \cup A_{+-} \cup A_{-+}) + m^*(A_{--})$$

□

d) if $E_1 \cdots E_N$ are measurable, then $\bigcup_{j=1}^N E_j$ and $\bigcap_{j=1}^N E_j$ are also measurable.

Pf by induction on N:

base case $N=2$: Both are true by Lemma 7.4.4 c)

Induction step: $K = N+1$

Assuming the proposition holds for N, we investigate $N+1$

We first want to show that

$$m^*(A) = m^*(A \cap (\bigcap_{j=1}^K E_j)) + m^*(A/(\bigcap_{j=1}^K E_j))$$

we can write

$$\bigcap_{j=1}^K E_j = \bigcap_{j=1}^{N+1} E_j = (\bigcap_{j=1}^N E_j) \cap E_{N+1} \quad - \text{ Let } E^* = \bigcap_{j=1}^N E_j$$
$$= E^* \cap E_{N+1}$$

Then by using te same argument as in c), split into disjoint sets and derive that the intersection of E^* and E_{N+1} is measurable. $\Rightarrow \bigcap_{j=1}^N E_j$ is measurable.

Now show

$$m^*(A) = m^*(A \cap (\bigcup_{j=1}^K E_j)) + m^*(A/(\bigcup_{j=1}^K E_j))$$

we again write

$$\bigcup_{j=1}^K E_j = \bigcup_{j=1}^{N+1} E_j = (\bigcup_{j=1}^N E_j) \cup E_{N+1} = E^* \cup E_{N+1} \quad \text{with } E^* = \bigcup_{j=1}^N E_j$$

and since the union of two measurable sets are measurable, $\bigcup_{j=1}^K E_j$ is measurable

□

e) Prove that every open or closed box is measurable

Pf: using interval notation for all boxes in $\mathbb{R}^n$ for simplicity, we can write

$$[a,b] = (-\infty,b] \cap [a,\infty)$$
$$= (-\infty,0]+b \cap a+[0,\infty)$$

and we know that translations and unions are measurable by Lemma 7.4.4 b), d). Thus the closed box $[a,b]$ is measurable. For the open box the same argument is proper.

NOTE: Half spaces are measurable by Lemma 7.4.2

□

f) Prove that if $m^*(E)=0$, then E is measurable:

Pf: We w.t.s. $m^*(A) = m^*(A\cap E) + m^*(A/E)$.

We know that

$m^*(A\cap E) \leq m^*(E) = 0 \iff m^*(A\cap E) = 0$ since m^* is non-negative

Since $A/E \subseteq A$, we have

$$m^*(A) \geq m^*(A/E) = m^*(A\cap E) + m^*(A/E)$$

Now, since $(A/E)\cup(A\cap E) = A$, by subadditivity we have

$$m^*(A) \leq m^*(A\cap E) + m^*(A/E)$$

And thus E is measurable

□