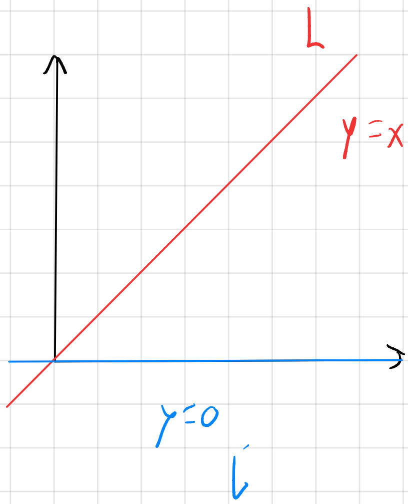
Hw 3

Ex 3: We wts. that a line $y=x$ has measure zero.

Pf: Consider the line in the plot of $y=x$



Let $L=\{(x,y): y=x\}$. The linear map

$$A = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$$

can be used with Pugh 6.3.9 to take L to $L'=\{(x,y): y=0\}$ by rotating it by $\frac{\pi}{4}$.

We know by Pugh 6.3.9 that

$m(L) = m(L')$

And we have previously seen that we can cover L' by countably many boxes like this:

For an $\varepsilon > 0$ let $B_i = (-2^i, 2^i) \times \left(-\frac{\varepsilon}{2^{2i+2}}, \frac{\varepsilon}{2^{2i+2}}\right)$. Så ser vi at

$$|B_i| = |2 \cdot 2^i| \cdot \left|2\frac{\varepsilon}{2^{2i+2}}\right| = 2^{i+1}\frac{\varepsilon}{2^{2n+1}} = \frac{\varepsilon}{2^i}$$

and since $\bigcup_i^\infty |B_i|$ covers L',

$$m(L') \le m\left(\bigcup_i^\infty |B_i|\right) \le \varepsilon \qquad \text{(Sub-additivity)}$$

and as $\varepsilon \to 0$ we find that $m(L') = 0 \iff m(L) = 0$

Ex 6

Thm 16

Using Pugh lemma 28, we can Separate E into a countable union of disjoint open cubes and a zero set, Let these open cubes be B_i. then $E = \bigsqcup_i B_i \cup Z$.

For each cube B_i. We have a Box surrounding it called A_i, decreasing and open sequences U_{ni}, V_{ni} and increasing and closed sequence $K_{ni} = A_i \setminus V_{ni}$. Since each $F_i = \bigcup_n K_{ni}$ is an F_σ-set contained in B_i and each $G_j = \bigcap_n U_{ni}$ is a G_δ-set containing B_i.

~~Since~~ Let $F = \bigcup_i F_i$ and $G_i = \bigcup_i G_i$. These are F_σ and G_δ sets and Thus $F \subset E \subset G$ furthermore $m(G \setminus F) = 0$ as

$m(\bigcup_i G_i \setminus \bigcup_i F_i) = \sum_i m(G_i \setminus F_i) = 0$ disjointness and additivity

The other way holds by the proof in Pugh.

$\square$

Thm 21 - unbounded and n-dimensional case

The n-dimensional case holds by the same proof as Pugh if $A \subset I^n$, $B \subset I^k$ where I^n and I^k are unit boxes

Assume now that either A or B (or both) are unbounded. Let the unbounded set be A. A can be written as a countable union of disjoint open boxes plus a zero set:

$$A = \bigsqcup_j \bar{A}_j \cup Z$$

If B is unbounded, let $B = \bigsqcup_i \bar{B}_i \cup Z'$

We can now describe $m(A) = \sum_j m(\bar{A}_j) + m(Z) = \sum_j m(\bar{A}_j)$ and $m(B) = \sum_i m(\bar{B}_i)$. Furthermore, $A \times B = \bigsqcup_j \bigsqcup_i \bar{A}_j \times \bar{B}_i$

we know that $m(\bar{A}_j \times \bar{B}_i) = m(\bar{A}_j)\, m(\bar{B}_i)$, so

$$m\Big(\bigsqcup_j \bigsqcup_i (\bar{A}_j \times \bar{B}_i)\Big) = \sum_i \sum_j m(\bar{A}_j)\, m(\bar{B}_i) = \sum_j m(\bar{A}_j) \sum_i m(B_i) = m(A)\, m(B)$$

So $m(A \times B) = m(A)\, m(B)$

$\square$

Ex 3:

Prove $J^*(A) = J^*(\bar{A}) = m(A)$.

We know that for a bounded set $A \subset R$:

$J^*(A) = \inf\{ \sum_{i=1}^{n} |I_i| : I_i \text{ open interval and } A \subset \bigcup_{i=1}^{n} I_i \}$

as $\bar{\ }$ means the closure of a set, $\bar{I}_i$ are closed intervals.

Pf: $J^*(A) \leq J^*(\bar{A})$

This holds by monotonicity since $A \subset \bar{A}$.

$J^*(A) \geq J^*(\bar{A})$

for each $\varepsilon > 0$ we can cover $\bar{I}_i = [a, b]$ by a

$\tilde{I}_i = (a-\varepsilon, b+\varepsilon)$. Thus

$\bigcup \bar{I}_i \subset \bigcup \tilde{I}_i$ and $J^*(\bar{A}) \leq J^*(\tilde{A})$ for $\tilde{A} \subset \bigcup_j \tilde{I}_j$

letting $\varepsilon \to 0$ we get $J^*(\tilde{A}) = J^*(A)$, so

$J^*(A) \geq J^*(\bar{A})$.

Thus $J^*(A) = J^*(\bar{A})$

By property 5 of Pugh ex 11, if A is compact, then $J^*(A) = m(A)$. A is bounded, so $\bar{A}$ is compact and $J^*(\bar{A}) = m(\bar{A})$, Thus $J^*(A) = J^*(\bar{A}) = m(\bar{A})$

$\square$