Using def 2: a subset E is measurable if

$\forall \varepsilon > 0 \; \exists U \supset E : m^*(U \setminus E) < \varepsilon$. and U is open

Prove Lemma 1.2.5. finite additivity for seperate sets

wts that for $E, F \subset \mathbb{R}^d$, $dist(E, F) > 0$ with

$$d(E, F) = \inf \{ |x - y| : x \in E, y \in F \}$$

then $m^*(E \cup F) = m^*(E) + m^*(F)$

Pf: $(\leq)$ by sub-additivity this holds.

$(\geq)$

We can utiliz an $\varepsilon > 0$. Let $\{A_i\}$ cover E and $\{B_j\}$ cover F. then $m^*(E) \leq \sum_i |A_i|$ and $m^*(F) \leq \sum_j |B_j|$.

Now let $\{C_k\}$ cover $E \cup F$. for an $\varepsilon > 0$ there must exist a cover s.t.

$$\sum_k |C_k| \leq m^*(E \cup F) + \varepsilon.$$

Suppose that $\{A_i\}$ is the part of $\{C_k\}$ that covers E, and $\{B_j\}$ the other part. then

$$m^*(E) + m^*(F) \leq \sum_k |C_k| \leq m^*(E \cup F) + \varepsilon.$$

and letting $\varepsilon \to$ we find that

$$m^*(E) + m^*(F) \leq m^*(E \cup F)$$

$\square$

Lemma 1: $m^*(A) = \inf\{m^*(U) \mid U \supset A, U \text{ is open}\}$

Pf:

($\leq$) by monotonicity this holds

($\geq$) By def 2 A is measurable, $\forall \varepsilon > 0 \; \exists U \supset A$, U open s.t.

$m^*(U \setminus E) < \varepsilon$

Definition of $m^*(E) = \inf\{\sum_i |B_i| : \{B_i\} \text{ covers } E\}$

Let $\varepsilon' > 0$ be such that

$\sum_i |B_i| \leq m^*(E) + \varepsilon'$

Since $m^*(U \setminus E)$ can be arbitrarily small, it can be such that

$m^*(U) \leq \sum_i |B_i| \leq m^*(E) + \varepsilon'$ *

By definition 2 we can pick appropriate U based on ε and ε'. As we let $\varepsilon \to 0$ we find that

$\inf\{m^*(U) \mid U \supset E, U \text{ open}\} \leq m^*(E) + \varepsilon$

and by then letting $\varepsilon' \to 0$

$\inf\{m^*(U) \mid U \supset E, U \text{ open}\} \leq m^*(E)$

$\square$

Lemma 2: if $E_1 \cdots$ is measurable, then $\bigcup_i E_i$ is measurable.

Pf: For each E_i there exist an open $V_i \supset E_i$ such that for $\varepsilon' > 0$, $m^*(V_i \setminus E_i) < \varepsilon'$. Now let $\varepsilon' = \frac{\varepsilon}{2^i}$ for any $\varepsilon > 0$. By sub-additivity,

$$m^*\left(\bigcup_i^\infty (V_i \setminus E_i)\right) = m^*\left(\bigcup_i^\infty V_i \setminus \bigcup_i E_i\right) \le \sum_i m^*(V_i \setminus E_i)$$

meaning

$$m^*\left(\bigcup_i^\infty V_i \setminus \bigcup_i^\infty E_i\right) \le \sum_i^\infty m^*(V_i \setminus E_i) < \sum_i^\infty \frac{\varepsilon}{2^i} = \varepsilon$$

Since $\bigcup_i^\infty V_i$ is open, $\bigcup_i E_i$ is measurable.

Lemma 4