Hw 12

This is, like Jianzhi, based on the notes from purdue.
Shout out!

We want to show that a closed 2-form on $\mathbb{R}^3$ is exact, i.e. for

$$w = f\, dx \wedge dy + g\, dz \wedge dx + h\, dz \wedge dy$$

if $dw = 0$, then there exists an α s.t. $d\alpha = w$.

assuming $dw = 0$, we find like with the 1-form on $\mathbb{R}^2$ that

$$dw = f_z\, dz \wedge dx \wedge dy + g_y\, dy \wedge dz \wedge dx + h_x\, dx \wedge dz \wedge dy$$

$$= (f_z + g_y - h_x)\, dx \wedge dy \wedge dz = 0$$

$$\Rightarrow f_z + g_y - h_x = 0 \iff f_z = h_x - g_y \quad *$$

We now define the following homotopy operation:

$$H_z(w) = \left(\int_0^z g(x,y,z)\,dz\right) dx + \left(\int_0^z h(x,y,z)\,dz\right) dy$$

Taking the exterior derivative and using $*$ we find

$$dH_z(w) = \left(\int_0^z g_y\, dz\right) dy \wedge dx + \left(\int_0^z g_z\, dz\right) dz \wedge dx$$

$$+ \left(\int_0^z h_x\, dz\right) dx \wedge dy + \int_0^z h_z\, dz)\, dz \wedge dy$$

By FTC $$= \int_0^z (h_z - g_y)\, dx \wedge dy + [g(x,y,z) - g(x,y,0)]\, dz \wedge dy$$

$$+ [h(x,y,z) - h(x,y,0)]\, dz \wedge dx$$

by $*$ $$= \int_0^z (f_z)\, dx \wedge dy + [g(x,y,z) - g(x,y,0)]\, dz \wedge dy$$

$$+ [h(x,y,z) - h(x,y,0)]\, dz \wedge dx$$

cont

by FTC $= [f(x,y,z) - f(x,y,0)] dx \wedge dy + [g(x,y,z) - g(x,y,0)] dz \wedge dy$

$+ [h(x,y,z) - h(x,y,0)] dz \wedge dx$

$= w - w(x,y,0)$ ← short hand notation

so we have $dH_z(w) = w - w(x,y,0)$ and

$d^2 H_z(w) = = dw - dw(x,y,0) = 0$

and since $dw = 0$ $dw(x,y,0) = 0 \Rightarrow w(x,y,0)$ is closed.

Furthermore, if we can find an α_1 s.t. $d\alpha_1 = w(x,y,0)$, then

$dH_z(w) = w - d\alpha_1 \Leftrightarrow d(H_z(w) + \alpha_1) = w$

Using the homotopy operation on $w(x,y,0)$ w.r.t. y and

x after each other we get

$dH_y(w(x,y,0)) = w(x,y,0) - w(x,0,0)$

$dH_x(w(x,0,0)) = w(x,0,0) - w(0,0,0)$

and using the same argument as previously we see that

① if $\exists \alpha_2 : d\alpha_2 = w(x,0,0) \Rightarrow w = d(H_z(w) + H_y(w(x,y,0)) + \alpha_2)$

② if $\exists \alpha_3 : d\alpha_3 = w(0,0,0) \Rightarrow w = d(H_z(w) + H_y(w(x,y,0)) + H_x(w(x,0,0)) + \alpha_3)$

③ $w(x,0,0)$ and $w(0,0,0)$ are closed

The goal is now to find a α_3 s.t. $d\alpha_3 = w(0,0,0)$

Observe that $w(0,0,0)$ depends on no variables and is thus a 2-form with constant weights. This can be expressed in the form of

$w(0,0,0) = a\, dx \wedge dy + b\, dz \wedge dy + c\, dy \wedge dz$

let $\alpha_3 = ax\,dy + bz\,dy + cy\,dz$ be a 1-form on $\mathbb{R}^3$. Then

$d\alpha_3 = w(0,0,0)$ and thus w is exact.