Hw 5

Tao 8.2.9

we start by noticing that for the set

$\{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p}$ for infinitely many integers $a, q\}$

it is obvious that we can limit a to $a \geq 0$ and q to $q \geq 0$ since if the condition is false for $a \geq 0$, $q > 0$, then it is false for $(-a, q)$, $(a, -q)$ and $(-a, -q)$. Using the same argument for the relationship between a and q we find that we only have to consider $0 \leq a \leq q$.

for an a, q, let $S_{a,q} = \{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p}\}$. This is $[0,1]$ if $\frac{c}{q^p} \geq 1$ and $[\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}]$. This is an interval:

$S_{a,q} = [0,1] \cap [\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}]$. by monotonicity, since

$S_{a,q} \subseteq [\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p}] = \tilde{S}_{a,q}$

we have

$$m(S_{a,q}) \leq m(\tilde{S}_{a,q}) = 2\frac{c}{q^p}$$

We now find the measure of $\sum_{q=1}^{\infty} \sum_{a=1}^{q} m(S_{a,q})$ by

$$\sum_{q=1}^{\infty} \sum_{a=1}^{q} m(S_{a,q}) \leq \sum_{q=1}^{\infty} \sum_{a=1}^{q} 2\frac{c}{q^p} = 2c \sum_{q=1}^{\infty} q \frac{1}{q^p} = 2c \sum_{q=1}^{\infty} \frac{1}{q^{p-1}}$$

and since $p > 2$, $2c \sum_{q=1}^{\infty} \frac{1}{q^{p-1}}$ converges and thus

$\sum_{q=1}^{\infty} \sum_{a=1}^{q} m(S_{a,q})$ has finite measure. By Borel-cantelli, the

set $\{x \in [0,1] : |x - \frac{a}{q}| \leq \frac{c}{q^p}$ for infinitely many integers $a, q\}$

has measure zero

$\square$

Exercise 8.2.9

We start by noting that $\int_R f_n \leq \frac{1}{4^n}$, so on R f_n is measurable and $m(f(R)) \leq \frac{1}{4^n}$. For an $\varepsilon > 0$ and $n \in \mathbb{N}$ let

$$X = \{x \in R : f_n(x) > \frac{1}{\varepsilon 2^n}\}.$$

Then $X_n = f_n^{-1}((\frac{1}{\varepsilon 2^n}, +\infty))$ and $\int_R f_n(X_n) = m(f_n(X_n)) \leq \int_R f_n \leq \frac{1}{4^n}$.

Got stuck here. Credit for next step goes to Shuqi Ke!

Define

$$S_n = \begin{cases} \frac{1}{\varepsilon 2^n} & x \in X_n \\ 0 & x \notin X_n \end{cases}$$

for all $x \in X_n$ $S_n \leq f_n$ and

$$\int_R S_n = m(X_n) \cdot \frac{1}{\varepsilon 2^n} \leq \frac{1}{4^n} \iff m(X_n) \leq \frac{\varepsilon}{2^n}.$$

Now we have shown that for an $\varepsilon > 0$, $\forall n \geq 1$

$$m(\{x \in R : f_n(x) > \frac{1}{\varepsilon 2^n}\}) \leq \frac{\varepsilon}{2^n}.$$

Now consider $X = \bigcup X_n$ for each $\varepsilon > 0$. We know that X_n is measurable, so $\bigcup X_n$ is measurable, and by subadditivity,

$$m(X) \leq \sum_{i=1}^{\infty} m(X_i) \leq \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = \varepsilon$$

We now have that for $\varepsilon > 0$, $n \geq 1$ there exists X st $m(X) < \varepsilon$. Lets now look at $R \setminus X$. For all $\bar{x} \in R \setminus X$, because $\bar{x} \notin X$, there exists an $N \geq 1$ s.t. for all $n \geq N$ $f_n(\bar{x}) \leq \frac{1}{\varepsilon 2^n} < \varepsilon$, and with the same N,

$$|f_n(\bar{x}) - 0| = f_n(x) < \varepsilon.$$

Thus $f_n(x)$ converges pointwise to 0 for $x \in R \setminus X$