Exercise 39

We have

① f, g measurable

② f^2, g^2 integrable.

By ①, and Exercise 28, $f \cdot g$ is measurable and thus $\int fg$ exist. By ② $\int f^2 \int g^2$ exists and

$$\int f^2 \int g^2 = \int |f|^2 \int |g|^2$$

By the Cauchy-Schwartz inequality* for integrals we find

$$\int |f|^2 \int |g|^2 \geq |\int fg \int fg| = |\int fg|^2$$

and thus

$$\sqrt{\int f^2 \int g^2} = \sqrt{\int |f|^2 \int |g|^2} \geq \int |fg|$$

$$= \int fg^+ + \int fg^- \geq \int fg^+ - \int fg^- \geq \int fg$$

* works by prop 38. Only thing left to show is that $\int fg$ for f, g is an inner product: $\langle f, g \rangle = \int fg$

Lemma: for f, g, measurable, $\langle f, g \rangle = \int fg$ is inner prod.

Pf:

Conjugate symmetry:

$$\int fg = \int gf \quad \text{since } fg = gf$$

linear mapping:

$$\int (af + bg) h = \int afh + \int bgh = a \int fh + b \int gh$$

Positive definiteness:

$$\int ff = \int f^2 = \int |f|^2 > 0 \quad \text{since } |f| \text{ is non-negative.}$$

$\square$

Exercise 48.

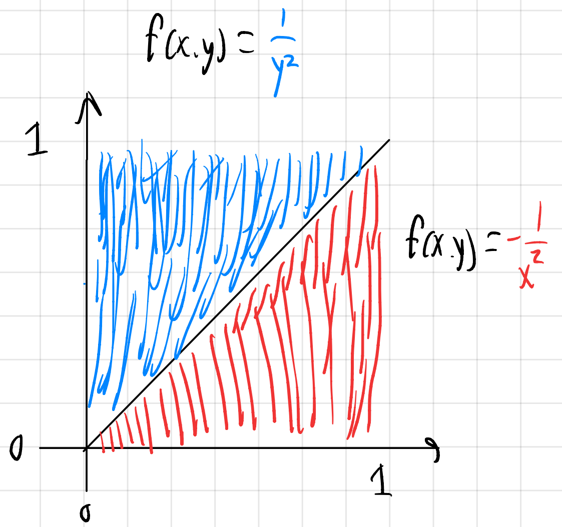
To be honest I found this very difficult in the sence of just wrapping my head around it was hard.
I also had difficulties with the other questions which took time away from this. I will probably return to this, if it is alright.

Exercise 5 3

a)
$$f(x,y) = \begin{cases} \frac{1}{y^2} & 0 < x < y < 1 \\ -\frac{1}{x^2} & 0 < y < x < 1 \end{cases}$$

Let us start by integrating iteratively. We use Pugh Corollary 70 which states that If Riemann integral exists, then it is equal to Lebesque integral (Thanks Griffin!). I also include Michaels plot, as it is a very intuitive way of looking at the function.

We start with $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$.



$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy$$
$$= \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy$$

now

$$\int_0^1 f(x,y)\,dx = \int_0^y f(x,y)\,dx + \int_y^1 f(x,y)\,dx$$
$$= \int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx = \left[\frac{1}{y^2} x \right]_0^y + \left[\frac{1}{x} \right]_y^1$$
$$= \frac{1}{y} + 1 - \frac{1}{y} = 1$$

So $\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = \int_0^1 1\,dy = [y]_0^1 = 1$

In the same way we find

$$\int_0^1 f(x,y)\,dy = \int_0^x f(x,y)\,dx + \int_x^1 f(x,y)\,dx$$
$$= \int_0^x -\frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dx = \left[-\frac{1}{x^2} y \right]_0^x + \left[-\frac{1}{y} \right]_y^1$$
$$= -\frac{1}{y} - 1 + \frac{1}{y} = -1$$

so

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = \int_0^1 -1\,dx = [-x]_0^1 = -1$$

Now for the double integral. We know that since the above integrals aren't equal, the Riemann double integral doesn't exist.

b) This does not break corollary 43 because f is negative and the corollary requires our fcns to be non-negative.

Exercise 5.8

a) Since E is measurable, for almost all $x \in E$, x is a density point, Thus for a decreasing sequence of boxes $\{Q_i\}$,

$$\lim_{Q_i \downarrow x} \frac{m(Q_i \cap E)}{m(Q_i)} = 1.$$

now for each box, make a ball that contains that box and is centered at p.

c) Take $E = [0, 1] \subset \mathbb{R}$. for an $\alpha \in [0, 1]$, let

$$x^- = \begin{cases} \frac{1-\alpha}{\alpha} & \alpha < \frac{1}{2} \\ 1 & \alpha \geq \frac{1}{2} \vee \alpha = 0 \\ 0 & \alpha = 1 \end{cases} \qquad x^+ = \begin{cases} \frac{\alpha}{1-\alpha} & \alpha > \frac{1}{2} \\ 1 & \alpha \leq \frac{1}{2} \vee \alpha = 1 \\ 0 & \alpha = 0 \end{cases}$$

Let $Q_n = \left[-\frac{x^-}{n}, \frac{x^+}{n}\right]$. Then $E \cap Q_n = \left[0, \frac{x^+}{n}\right]$. Let $p = 0$. then if $\alpha = 0$ trivially $\delta(p, E) = 0$ and if $\alpha = 1$ $\delta(p, E) = 1$. if $\alpha = \frac{1}{2}$ then

$$\lim_{Q_n \downarrow p} \frac{m(Q_n \cap E)}{m(Q_n)} = \lim_{n \to \infty} \frac{1/n}{2/n} = \frac{1}{2}$$

for $1 > \alpha > \frac{1}{2}$

$$\lim_{Q_n \downarrow p} \frac{m(Q_n \cap E)}{m(Q_n)} = \lim_{n \to \infty} \frac{\frac{\alpha}{(1-\alpha)n}}{\frac{\alpha}{(1-\alpha)n} + \frac{1}{n}} = \lim_{n \to \infty} \frac{\frac{\alpha}{(1-\alpha)n}}{\frac{1}{(1-\alpha)n}} = \alpha$$

for $0 < \alpha < \frac{1}{2}$

$$\lim_{Q_n \downarrow p} \frac{m(Q_n \cap E)}{m(Q_n)} = \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{n} + \frac{(1-\alpha)}{n\alpha}} = \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{1}{n\alpha}} = \alpha$$

b) My ultimately unsuccesful try at a covering was to define $E_n = [\frac{1}{2^n}, \frac{\alpha+1}{2^n}]$ st that each E_i was defined to be segments of $[\frac{1}{2^n}, \frac{1}{2^{n+1}}]$ of length $\alpha/2^n$.

This can be seen in the line →

By placing a ball with center 0 and radius $\frac{1}{2^n}$: $B_{\frac{1}{2^n}}(0)$

My goal was to somehow show that for

$$E = \bigcup_{i=0}^{\infty} E_i, \quad \bar{E}_N = E \setminus \left(\bigcup_i^N E_i\right)$$

$$\lim_{r\to 0} \frac{m(B_r(P)\cap E)}{m(B_r(P))} = \lim_{n\to\infty} \frac{m(B_{1/2^n}(P)\cap E)}{m(B_{1/2^n}(P))} = \lim_{n\to\infty} \frac{m([-\frac{1}{2^n}, \frac{1}{2^n}]\cap \bar{E}_n)}{m([-\frac{1}{2^n}, \frac{1}{2^n}])}$$

My intuition says that this will work as there an over lap wher th numerator has an infinite sum as measure

$$\sum_{n=1}^{\infty} \frac{\alpha}{2^n} = \alpha \sum_{k=n}^{\infty} \frac{1}{2^k}$$

However I couldnt make the calculations work.

d) Due to me not finishing c i didn't get a chance to answer this, Michael did however share a lenghty paper on this, which i might read in order to understand this further.

Exercise 66.

Example used from Counter-examples.com by Jean-Pierre Marx.

Let $P = \sum P_n$ be a convergent series of positive numbers, e.g. $P_n = \frac{1}{n^2}$. Let $f: [0,1] \to \mathbb{R}$ be defined as

$$f(x) = \sum_{d_n \le x} \frac{1}{n^2}$$

for a $d_n \in \mathbb{Q} \cap [0,1]$, $n \in \mathbb{N}$, i.e. d_n is the nth element in the countable set $\mathbb{Q} \cap [0,1]$.

for a $0 \le x < y \le 1$.

$$f(y) - f(x) = \sum_{x < d_n \le y} \frac{1}{n^2} > 0$$

since $\forall n \; \frac{1}{n^2} > 0$ and there always exists a $d_n : x < d_n \le y$.

Thus f is monotone.

Lemma 1: f is right cont on $[0,1]$

Pf: pick $x \in [a,b]$. For any $\varepsilon > 0$ there exist $N \in \mathbb{N}$ st

$$0 < \sum_{n \ge N} \frac{1}{n^2} < \varepsilon$$

Let $\delta > 0$ be small enough that no point $\{\frac{1}{1} \cdots \frac{1}{N^2}\}$ in $(x, x+\delta)$. Then

$$0 < f(y) - f(x) \le \sum_{n \ge N} \frac{1}{n^2} < \varepsilon \quad \text{for } y - x < \delta$$

with $y \in (x, x+\delta)$ $\square$

Lemma 2: f is discont on $\mathbb{Q} \cap [0,1]$

Pf: Take a $x = d_m \in \mathbb{Q} \cap [0,1]$, for any $0 \le y < x$

$$f(x) - f(y) = \sum_{y < d_n \le x} \frac{1}{n^2} > \frac{1}{m^2}$$

so it is not left cont at $x \in \mathbb{Q}_n \cap [0,1]$ $\square$

Lemma 3. f is cont at all $x \in [0,1] \setminus \mathbb{Q}$.

Pf. Let $N(x) := \{ n \in \mathbb{N} \mid d_n < x \}$. Since $x \notin \mathbb{Q} \cap [0,1]$,

$$f(x) = \sum_{n \in N(x)} \frac{1}{n^2}$$

for any $\varepsilon > 0$ there exists a finite subset $N_0 \in N(x)$ st

$$\sum_{n \in N_0} \frac{1}{n^2} > f(x) - \varepsilon$$

for any $\delta > 0$ st the interval $(x - \delta, x)$ doesnt contain any $\frac{1}{n^2}$ for $n \in N_0$ we have for $y \in (x - \delta, x)$

$$f(x) - \varepsilon \le f(y) = \sum_{d_n \le y} \frac{1}{n^2} < f(x)$$

and thus $f(x)$ is cont on $x \notin \mathbb{Q} \cap [0,1]$