Hw 10

12)

we are given

$f_1(s,t) = (b + a\cos s)\cos t$

$f_2(s,t) = (b + a\cos s)\sin t$

$f_3(s,t) = a\sin s$

with

$f(s,t) = (f_1(s,t), f_2(s,t), f_3(s,t))$

Since $\sup\{\cos\} = \sup\{\sin\} = 1$, $\inf\{\cos\} = \inf\{\sin\} = -1$

The range of K is $([-b-a, b+a], [-b-a, b+a], [-a, a])$

a)

we first find that

$$\nabla f_1(s,t) = (-a\sin s\cos t, -(b+a\cos s)\sin t)$$

This is 0 when

① $a\sin s\cos t = 0$

② $(b+a\cos s)\sin t = 0$

for ① This happens when $\sin s = 0$ or $\cos t = 0$

and for ② when $\sin t = 0$. When $\sin t = 0$, $\cos t \neq 0$, so

for ① $\sin s = 0$ is the only option. These hold when

$t = 0$ or $t = \pi$, $s = 0$ or $s = \pi$, for all those values

$f_2(s,t) = f_3(s,t) = 0$ fully, the points are

$$P_1 = \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} b-a \\ 0 \\ 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} -b+a \\ 0 \\ 0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} -b-a \\ 0 \\ 0 \end{bmatrix}$$

b)

we first find that

$$\nabla f_3(s,t) = (a\cos s,\ 0)$$

Thus $\nabla_3 f(s,t) = 0$ when $\cos s = 0$ so $s' = (\frac{1}{2}+k)\pi$ $k \in \mathbb{N}$

for all s' $f_3(s,t) = 0$. Since t is still variable,

$f_2(s,t)$ and $f_1(s,t)$ can have all values in the range

Thus for all $q \in [-b\cos t, b\cos t] \times [-b\sin t, b\sin t] \times \{-a, a\}$,

$t \in [0, 2k\pi)$, $\nabla f_3(f^{-1}(q)) = 0$

c)

we have

$$f_1(s,t) = (b + a\cos s)\cos t$$

$$\nabla f_1(s,t) = (-a\sin s\cos t,\ -(b+a\cos s)\sin t)$$

we found the points

$$P_1 = \begin{bmatrix} b+a \\ 0 \\ 0 \end{bmatrix}, P_2 = \begin{bmatrix} b-a \\ 0 \\ 0 \end{bmatrix} \quad P_3 = \begin{bmatrix} -b+a \\ 0 \\ 0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} -b-a \\ 0 \\ 0 \end{bmatrix}$$

At these points we have $t=0$ or $t=\pi$, $s=0$ or $s=\pi$

at $t=0$ $\cos t$ will only decrease with a change in t.

at $t=\pi$ it will only increase. For both, sin changes in the same direction as t. This also holds for $s=0$ and $s=\pi$. Given $b>a>0$, so $f(0,0) > f(0,\pi)$ and $f(\pi,0) < f(\pi,\pi)$. By the above, sin will flip sign, so $f(0,0)$ is a local maximum $f(\pi,\pi)$ is a local minimum and $f(\pi,0)$ and $f(0,\pi)$ are saddle points.

Since The points are on the boundary only they need to be considered (Maximum Value Theorem)

13)

Let $f: \mathbb{R}^1 \to \mathbb{R}^3$ be differentiable, and $|f(t)| = 1 \quad \forall t$.

$$h(t) = f(t) f(t) = f(t)^2 = |f(t)|^2 = 1$$

Then

$$h'(t) = f(t) f'(t) + f'(t) f(t) = 0$$

so $f'(t) f(t) = 0$

Geometrically since $|f(t)| = \sqrt{f_1(t)^2 + f_2(t)^2 + f_3(t)^2}$ we would we can think of $f(t)$ as mapping a value to a point on a unit sphere in 3D space. Since f is diff $\Rightarrow$ cont this mapping is a continuous curve on the surface. Think of $f'(t)$ as the normal vector to a point on the sphere $f(t)$.

14)

We have the equations.

① $3x + y - z + w^2 = 0$

② $x - y + 2z + w = 0$

③ $2x + 2y - 3z + 2w = 0$

We can write this as

$$f_1(x, y, z, w) = 3x + y - z + w^2$$

$$f_2(x, y, z, w) = x - y + 2z + w$$

$$f_3(x, y, z, w) = 2x + 2y - 3z + 2w$$

with $f = (f_1, f_2, f_3)$. We now wish to solve for $f(x, y, z, w) = 0$ in terms of different variables for this the implicit

$x + y + z^2 = 0$

$2x + 3y - z = 0$

$y = -2x + z$

$x - 2x + z + z^2 = 0 = -3x + z + z^2$

$x = \frac{1}{3} z + \frac{1}{3} z^2$

$-3(\frac{1}{3} z + \frac{1}{3} z^2) + z + z^2 = 0$

function theorem comes in handy. By Rudin (I like his version better) we can solve $f(x,y,z,u)=(0,0,0,0)$ in terms of x,y,z, or u if conditions are met: Let the variables we solve for be h and let the term be k. Then if $A_h = A(h,0)$ is invertible we can write $f(x,y,z,u)$ as

$$f(x,y,z,u) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = Av \qquad v=(x,y,z,u)^T$$

we see that for $h=(x,y,u)$ $h=(x,z,u)$, and $h=(y,z,u)$ The submatrix A_h has non-zero determinant (Thank you Maple) for $u \neq \frac{3}{2}$. for $h=(x,y,z)$ $\det(A_h)=0$, So by the implicit function theorem we can not solve wrt. u, but we can for x,y,z

Pugh 14

Honestly i don't even know where to begin with this question... My guess was to use Characterization of diagonalizable matrizes by linear independent eigenvectors But even with this I do not know how to proceed. I will ask around and see if anyone has any ideas and correct my work if I get feedback

24) we have

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{else} \end{cases}$$

first of all $f(x,0) = f(y,0) = 0$, $\lim_{x \to 0} f(x,0) = \lim_{y \to 0} f(0,y) = 0$ and $f(0,0) = 0$

$$\lim_{t \to 0} f(th,t) = \lim \frac{t^2h^2(t^2h^2 - t^2)}{t^2h^2 + t^2} = \lim_{t \to 0} \frac{t^4h^4 - t^4h^2}{t^2(h^2+1)} = \lim_{t \to 0} \frac{t^2(h^4 - h^2)}{h^2+1} = 0$$

for $h \in \mathbb{R}$, So f is cont on $\mathbb{R}$. Now, since

$$\frac{\partial f}{\partial x} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} \qquad \frac{\partial f}{\partial y} = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2+y^2)^2} \qquad f'(0,0) = 0$$

$$\lim_{x \to 0} \frac{\partial f}{\partial x}(0,0) = \lim_{y \to 0} \frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} \frac{\partial f}{\partial x}(th,t) = \lim_{t \to 0} \frac{\partial f}{\partial y}(th,t) = 0$$

so Partial derivatives are continuous and differentiable, meaning second derivative exists, so partial ones do as well. But

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h \to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h \to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

so $\dfrac{\partial^2 f}{\partial x \partial y} \neq \dfrac{\partial^2 f}{\partial y \partial x}$.