Hw 11

1

- Consider the following forms on $\mathbb{R}^n \setminus 0$:
  - 2-form: $\Omega_1 = |x|^{-2}(x_1 dx_2 - x_2 dx_1)$
  - 3-form: $\Omega_2 = |x|^{-3}(x_1\, dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_2 \wedge dx_2)$

First we wts that $d\Omega_1 = 0$. We note that

$$|x|^{-2} = \sqrt{x_1^2 + x_2^2}^{-2} = \frac{1}{x_1^2 + x_2^2}$$

$$d\Omega_1 = d(|x|^{-2}(x_1 dx_2 - x_2 dx_1))$$

$$= d\left(\frac{1}{x_1^2+x_2^2}(x_1 dx_2 - x_2 dx_1)\right)$$

$$= d\left(\frac{x_1}{x_1^2+x_2^2}\right) \wedge dx_2 - d\left(\frac{x_2}{x_1^2+x_2^2}\right) \wedge dx_1$$

with $f(x,y) = \dfrac{x_1}{x_1^2+x_2^2}$, $g(x_1, x_2) = \dfrac{x_2}{x_1^2+x_2^2}$

$$= df \wedge dx_2 - dg \wedge dx_1$$

$$= \frac{\partial f}{\partial x_1} dx_1 \wedge dx_2 - \frac{\partial g}{\partial x_2} dx_2 \wedge dx_1$$

(Two terms cancel by $dx_i \wedge dx_i = 0$)

$$= \frac{\partial f}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial g}{\partial x_2} dx_1 \wedge dx_2$$

$$= \left(\frac{\partial f}{\partial x_1} + \frac{\partial g}{\partial x_2}\right) dx_1 \wedge dx_2$$

So it is enough to show that $\dfrac{\partial f}{\partial x_1} = -\dfrac{\partial g}{\partial x_2}$ $*$

$$\frac{\partial f}{\partial x_1} = \frac{x_1^2 + x_2^2 - 2x_1^2}{(x_1^2+x_2^2)^2} = \frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2}$$

$$\frac{\partial g}{\partial x_2} = \frac{x_1^2 + x_2^2 - 2x_2^2}{(x_1^2+x_2^2)^2} = -\frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2}$$

So $*$ is true and $d\Omega_1 = 0$

Next that $d\Omega_2 = 0$

$$\Omega_2 = |x|^{-3}(x_1\, dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2)$$

$$= \frac{x_1}{(x_1^2+x_2^2+x_3^2)^{3/2}}\, dx_2 \wedge dx_3 - \frac{x_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}\, dx_1 \wedge dx_3$$

$$+ \frac{x_3}{(x_1^2+x_2^2+x_3^2)^{3/2}}\, dx_1 \wedge dx_2$$

Using the same argument as previously, $d\Omega_2 = 0$ iff

$$(*)\quad \frac{\partial}{\partial x_1}\underbrace{\left(\frac{x_1}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right)}_{f(x_1,x_2,x_3)} + \frac{\partial}{\partial x_2}\underbrace{\left(\frac{x_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right)}_{g(x_1,x_2,x_3)} + \frac{\partial}{\partial x_3}\underbrace{\left(\frac{x_3}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right)}_{h(x_1,x_2,x_3)} = 0$$

$$\frac{\partial f}{\partial x_1} = \frac{-2x_1^2+x_2^2+x_3^2}{(x_1^2+x_2^2+x_3^2)^{5/2}} \qquad \frac{\partial g}{\partial x_2} = \frac{-2x_2^2+x_1^2+x_3^2}{(x_1^2+x_2^2+x_3^2)^{5/2}} \qquad \frac{\partial h}{\partial x_3} = \frac{-2x_3^2+x_1^2+x_2^2}{(x_1^2+x_2^2+x_3^2)^{5/2}}$$

so $(*)$ is true and $d\Omega_2 = 0$

General statement: $\Omega_{n-1} = |x|^{-n}\left(\sum_{i=1}^{n} (-1)^{i+1} x_i\, dx_{I-i}\right)$

$\Rightarrow d\Omega_{n-1} = 0$

uses notation from lecture

PF:

Let Ω_{n-1} be a n-form as described above

Then we rewrite

$$\Omega_{n-1} = \sum_{i=1}^{n}\left(\frac{(-1)^{i+1} x_i}{(\sum_{j=1}^{n} x_i^2)^{\frac{n}{2}}}\, d_{I-i}\right)$$

Then

$$d\Omega_{n-1} = \sum_{i=1}^{n} (-1)^{i+1}\, d\left(\frac{x_i}{(\sum_{j=1}^{n} x_j^2)^{n/2}}\right) \wedge dx_{I-i}$$

$$\ast \quad = \sum_{i=1}^{n} (-1)^{i+1} \frac{\partial}{\partial x_i}\left(\frac{x_i}{\left(\sum_{j=1}^{n} x_j^2\right)^{n/2}} \right) dx_i \wedge dx_{I-1}$$

The d_i is moved into the wedges $i-1$ times so they all line up, negating all terms $i-1$ times.

since $\frac{\partial}{\partial x_i}\left(\left(\sum_{j=1}^{n} x_j^2\right)^{n/2} \right) = n x_i \left(\sum_{j=1}^{n} x_j^2\right)^{n/2 - 1}$

$$\ast \quad = \sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{n} x_j^2\right)^{n/2} - n x_i^2 \left(\sum_{j=1}^{n} x_j^2\right)^{n/2-1}}{\left(\sum_{j=1}^{n} x_j^2\right)^{n}} dx_I$$

$$= \sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{n} x_j^2\right)^{n/2-1} \left(\sum_{j=1}^{n} x_j^2 - n x_i^2\right)}{\left(\sum_{j=1}^{n} x_j^2\right)^{n}}$$

$$= \frac{1}{\left(\sum_{j=1}^{n} x_j^2\right)^{n/2+1}} \underbrace{\left(n\left(\sum_{j=1}^{n} x_j^2\right) - n\left(\sum_{i=1}^{n} x_i^2\right)\right)}_{=0}$$

$$= 0$$

so $d\Omega_{n-1} = 0$

let $\gamma : [0,1]^2 \to \mathbb{R}^3$ be a 2-cell on $\mathbb{R}^3$ defined by

$$\gamma : (s,t) \mapsto (\sin(\pi s)\cos(2\pi t), \sin(\pi s)\sin(2\pi t), \cos \pi s)$$

what is $\int_\gamma \Omega_2$? Recall

$$\Omega_2 = |x|^{-3}(x_1\, dx_2 \wedge dx_3 - x_2\, dx_1 \wedge dx_3 + x_3\, dx_1 \wedge dx_2)$$

$$= \frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_2 \wedge dx_3 - \frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_1 \wedge dx_3$$

$$+ \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} dx_1 \wedge dx_2$$

since γ output coordinate to unit sphere, $|x| = 1$

$$\int_\gamma \Omega_2 = \int_\gamma x_1\, dx_2 \wedge dx_3 - x_2\, dx_1 \wedge dx_3 + x_3\, dx_1 \wedge dx_2$$

$$= \int_{t=0}^{1}\int_{s=0}^{1}\Bigg(\sin(\pi s)\cos(2\pi t)\det\begin{bmatrix} \frac{\partial \gamma_2}{\partial s} & \frac{\partial \gamma_2}{\partial t} \\ \frac{\partial \gamma_3}{\partial s} & \frac{\partial \gamma_3}{\partial t} \end{bmatrix}$$

$$- \sin(\pi s)\sin(2\pi t)\det\begin{bmatrix} \frac{\partial \gamma_1}{\partial s} & \frac{\partial \gamma_2}{\partial t} \\ \frac{\partial \gamma_1}{\partial s} & \frac{\partial \gamma_3}{\partial t} \end{bmatrix}$$

$$+ \cos(\pi s)\det\begin{bmatrix} \frac{\partial \gamma_1}{\partial s} & \frac{\partial \gamma_1}{\partial t} \\ \frac{\partial \gamma_2}{\partial s} & \frac{\partial \gamma_2}{\partial t} \end{bmatrix}\Bigg)\, ds\, dt$$

Now since

$$\frac{\partial \gamma_1}{\partial s} = \pi\cos(\pi s)\cos(2\pi t), \qquad \frac{\partial \gamma_1}{\partial t} = -2\pi\sin(\pi s)\sin(2\pi t)$$

$$\frac{\partial \gamma_2}{\partial s} = \pi\cos(\pi s)\sin(2\pi t), \qquad \frac{\partial \gamma_2}{\partial t} = 2\pi\sin(\pi s)\cos(2\pi t)$$

$$\frac{\partial \gamma_3}{\partial s} = -\pi\sin(\pi s), \qquad \frac{\partial \gamma_3}{\partial t} = 0$$

$$\det\begin{bmatrix} \frac{\partial \gamma_2}{\partial s} & \frac{\partial \gamma_2}{\partial t} \\ \frac{\partial \gamma_3}{\partial s} & \frac{\partial \gamma_3}{\partial t} \end{bmatrix} = \pi\cos(\pi s)\cos(2\pi t)\cdot 0 - 2\pi^2\sin(\pi s)^2\cos(2\pi t)$$

$$\det\begin{bmatrix} \frac{\partial \gamma_1}{\partial s} & \frac{\partial \gamma_1}{\partial t} \\ \frac{\partial \gamma_3}{\partial s} & \frac{\partial \gamma_3}{\partial t} \end{bmatrix} = 2\pi^2\sin(\pi s)^2\sin(2\pi t)$$

$$\det\begin{bmatrix} \frac{\partial \gamma_1}{\partial s} & \frac{\partial \gamma_1}{\partial t} \\ \frac{\partial \gamma_2}{\partial s} & \frac{\partial \gamma_2}{\partial t} \end{bmatrix} = 2\pi^2\sin(\pi s)\cos(\pi s)\sin(2\pi t)\cos(2\pi t)$$
$$+ 2\pi^2\cos(\pi s)\sin(\pi s)\sin(2\pi t)^2$$

$$\int_\gamma \Omega_2 = \int_{t=0}^{1}\int_{s=0}^{1}\Big(\sin(\pi s)\cos(2\pi t)\left(-2\pi^2 \sin(\pi s)^2 \cos(2\pi t)\right)$$

$$- \sin(\pi s)\sin(2\pi t)\left(2\pi^2 \sin(\pi s)^2 \sin(2\pi t)\right)$$

$$+ \cos(\pi s)\big(2\pi^2 \sin(\pi s)\cos(\pi s)\sin(2\pi t)\cos(2\pi t)$$

$$+ 2\pi^2 \cos(\pi s)\sin(\pi s)\sin(2\pi t)^2\big)\Big)\, ds\, dt$$

$$= \int_{t=0}^{1}\int_{s=0}^{1} 2\pi^2 \sin(\pi s)^3 + 2\pi^2 \cos(\pi s)^3 \sin(\pi s)\, ds\, dt$$

← Got stuck here, luckily Jianzhi found a way to simplify this. Good job

$$\overset{*}{=} 2\pi^2 \int_{t=0}^{1}\int_{s=0}^{1} \sin \pi s \, ds\, dt$$

$$= 2\pi^2 \int_{t=0}^{1} \left[-\frac{\cos(\pi s)}{\pi} \right]_0^1 dt$$

$$= 2\pi^2 \int_{t=0}^{1} \frac{1}{\pi}(1+1)\, dt$$

$$= 2\pi^2 \int_{t=0}^{1} \frac{2}{\pi}\, dt = 4\pi$$

$*$ $\cos(x)^2 + \sin(x)^2 = 1$

I have not been able to show this myself. My thoughts on this are, assuming the statement is true that somehow we could transform this into polar coordinates using sin, cos, I have not had any succes though, sadly. All out of luck, I read Jianzhi's solution, which was a great way of using the same method as before to reduce the term to

$$\int_{\gamma} \frac{4}{(1+a^2+b^2)^2}\, da\, db$$ and reducing to polar coordinates,

getting 4π.