Hw 6

Exercise 8.3.2

a) Since $f: \Omega \mapsto R$ is absolutely integrable,

$\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-$ where f^+, f^- are non-negative functions.

By 8.2.6 (a) we get

$$c\int_\Omega f = c\left(\int_\Omega f^+ - \int_\Omega f^-\right) = c\int_\Omega f^+ - c\int_\Omega f^- = \int_\Omega cf^+ - \int_\Omega cf^-$$

and since

$$cf^+ = \max(cf, 0) \qquad cf^- = -\min(cf, 0)$$

we find by Def 8.3.1 that

$$\int_\Omega cf^+ - \int_\Omega cf^- = \int_\Omega cf$$ note on page 3! $\square$

(b) Wts. $(f+g)$ is absolutely integrable with

$$\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$$

By def 8.3.1:

$$\int_\Omega f = \int_\Omega f^+ + \int_\Omega f^-, \qquad \int_\Omega g = \int_\Omega g^+ + \int_\Omega g^-$$

By Lemma 8.2.10

$$\int_\Omega f + \int_\Omega g$$
$$= \int_\Omega f^+ + \int_\Omega f^- + \int_\Omega g^+ + \int g^-$$
$$= \int_\Omega (f^+ + g^+) + \int_\Omega (f^- + g^-)$$

and since

$$f^+ + g^+ = \max(f, 0) + \max(g, 0) = \max(f+g, 0)$$

$$f^- + g^- = -\min(f+g, 0)$$ by the same argument

we have by def 8.3.1

$$\int_\Omega (f+g) = \int_\Omega (f^+ + g^+) + \int_\Omega (f^- + g^-) = \int_\Omega f + \int_\Omega g$$

And $f+g$ is absolutely integrable $\square$

(c) W.t.s. If $f(x) \le g(x)$ $\forall x \in \Omega$, then $\int_\Omega f \le \int_\Omega g$

Lets first use Def 8.3.1

$$f = f^+ - f^- \qquad\qquad g = g^+ - g^-$$

Where

$$f^+ = \max(f,0), \quad f^- = -\min(f,0)$$

$$g^+ = \max(g,0), \quad g^- = -\min(g,0)$$

If $f(x) \le g(x)$ $\forall x \in \Omega$, then

$f^+(x) \le g^+(x)$ Since $\max(f,0) \le \max(g,0)$

$f^-(x) \ge g^-(x)$ since $\min(f,0) \le \min(g,0)$

$\Leftrightarrow -\min(f,0) \ge -\min(g,0)$

By Def 8.3.2

$$\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-, \qquad \int_\Omega g = \int_\Omega g^+ - \int_\Omega g^-$$

and by proposition 8.2.6 (c) and

$$\int_\Omega f^+ \le \int_\Omega g^+ \quad \text{and} \quad \int_\Omega f^- \ge \int_\Omega g^-$$

meaning

$$\int f_\Omega = \int_\Omega f^+ - \int_\Omega f^- \le \int_\Omega g^+ - \int_\Omega g^- = \int_\Omega g$$ $\square$

(d) W.t.s if $f(x) = g(x)$ for a.e. $x \in \Omega$ then $\int_\Omega f = \int_\Omega g$

We start by using Def 8.3.1:

$$f = f^+ - f^-, \quad g = g^+ - g^-$$

By the definitions of f^+, f^-, g^+, g^- and using Prop 8.2.6(d) we find!

$$\int_\Omega f^+ = \int_\Omega g^+, \quad \int_\Omega f^- = \int_\Omega g^- \text{ a.e.} \Rightarrow \int_\Omega f = \int_\Omega g \text{ a.e.}$$ $\square$

Exercise 8.3.3

We know f, g absolutely integrable and $\int_\Omega f = \int_\Omega g$, and by Prop 8.3.1 $f(x) \leq g(x) \Rightarrow \int_\Omega f \leq \int_\Omega g$. Then by Prop 8.3.1

$$h(x) := g(x) - f(x) = g(x) + (-f(x)) \geq 0$$

is a non-negative function. Its integra exists and

$$\int_\Omega h(x) = \int_\Omega g - \int_\Omega f = 0$$

and since $h(x) \geq 0$, then $h(x) = 0$ a.e. as otherwise we would have $h(x) > 0$ for a non zero set and

$$\int_\Omega h(x) > 0$$

This is a contradiction, and $f(x) = g(x)$ a.e. $\square$

Note on Exercise 8.3.2 a)

proof only for $c > 0$. If $c = 0$, $cf^+ = 0$, $cf^- = 0$ and $cf = 0$

$$\int cf^+ - \int cf^- = 0 - 0 = 0 = \int cf$$

If $c < 0$, let $\bar{c} = -c$. then $cf^+ = c\max(f, 0) = -\min(\bar{c}f, 0)$ and $cf^- = -c\min(f, 0) = \max(\bar{c}f, 0)$ and

$$c\int_\Omega f^+ - c\int_\Omega f^- = -\int_\Omega \bar{c}f^+ + \int_\Omega \bar{c}f^- = \int_\Omega \bar{c}f^- - \int_\Omega \bar{c}f^+$$
$$= \int_\Omega \bar{c}(-f) = \int_\Omega cf$$

$\square$