Hw 8

2a) as in the original proof, define for $f: [a,b]^d \to \mathbb{R}$

$X(k,\ell) = \{ x \in [a,b]^d : \forall n \geq k \Rightarrow |f_n(x) - f(x)| < 1/\ell \}$

if we fix ℓ, then since $f_n(x) \to f(x)$ a.e. we have

$$\bigcup_k X(k,\ell) \cup Z(\ell) = [a,b]^d$$

where $Z(\ell)$ is a zero set.

We know that $m(X(k,\ell)) \to \prod(b_i - a_i)$ as $k \to \infty$, which is the measure of the domain of f_n. Thus we can define an increasing sequence of $\{k_i\}_{i \in \mathbb{N}}$ s.t. for $X_\ell = X(k_\ell, \ell)$ we have $m(X_\ell^c) < \varepsilon/2^\ell$. Thus

$$m\left(\bigcup_\ell^\infty X_\ell^c\right) = m(X^c) < \varepsilon, \quad \text{where } X = \bigcap_\ell^\infty X_\ell$$

if f_n converges uniformly on X, then given $\sigma > 0$ we fix ℓ s.t. $1/\ell < \sigma$, $\forall n \geq k_\ell$ we have

$$x \in X \Rightarrow x \in X_\ell = X(k_\ell, \ell) \Rightarrow |f_n(x) - f(x)| < 1/\ell < \sigma$$

and thus f_n converges uniformly towards f.

2b)

we can consider a sequence f_n on an unbounded domain. The argument works for all dimensions but will be shown in 1. From the proof of Egoroff's theorem, define $X(k,\ell) = \{x \in S : \forall n \geq k \ |f_n(x) - f(x)| < 1/\ell\}$ for unbounded S with finite measure. Then $m(X(k,\ell)) \to m(S)$ as $k \to \infty$ because $\bigcup_k^\infty X(k,\ell) \cup Z(\ell) = S$, where $Z(\ell)$ is a zero set. Then follows the original proof.

2c) Consider $f_n(x) = \begin{cases} 1 & x \in [n, n+1] \\ 0 & \text{otw.} \end{cases}$

we know that $f_n \to f$ a.e. but it is NOT a uniform conv.

2d) With the proof of Egoroff's theorem as a starting point, for an $\varepsilon > 0$ $K \cap S^c$ is bounded because $K \subset \mathbb{R}$ is compact. It thus has finite measure. With $S \cap K^c \subseteq S$ as replace-ment of X^c, and $K \cap S^c$ being X the proof follows from the proof of Erogoff's theorem.