Hw 9

Rudin 8.6

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & x, y \neq 0 \\ 0 & \text{else} \end{cases}$$

To Clarify we use notation $D_1 f = D_x f$ and $D_2 f = D_y f$ as we work in $\mathbb{R}^2$. $(D_x f)(x,y)$ is just the partial derivative wrt. x, and likewise $D_y f$, if they exist. They exist if the following limits exist:

$$(D_x f)(x,y) : \lim_{t \to 0} \frac{f(p+te_1) - f(p)}{t} = \lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t}$$

likewise for $(D_y f)(x,y)$: $\lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t}$

These are standard partial derivatives and we know that they exist:

$$(D_x f)(x,y) = \frac{\partial f}{\partial x} = \frac{y(x^2+y^2) - xy(2x)}{x^4+y^4+2x^2y^2}$$

$$(D_y f)(x,y) = \frac{\partial f}{\partial y} = \frac{x(x^2+y^2) - xy(2y)}{x^4+y^4+2x^2y^2}$$

These derivatives are defined everywhere but $(0,0)$. Here

$$D_x f(0,0) = \lim_{t \to 0} \frac{f(0+t,0) - f(0,0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0$$

$$D_y f(0,0) = \lim_{t \to 0} \frac{f(0,0+t) - f(0,0)}{t} = 0$$

so $D_x f$, $D_y f$ is defined on $\mathbb{R}^2$. But since

$$\lim_{t \to 0} f(t,t) = \lim_{t \to 0} \frac{t^2}{t^2+t^2} = \lim_{t \to 0} \frac{1}{2} = \frac{1}{2}$$

but $f(0,0)=0$ $f(x,y)$ is not cont at $f(0,0)$.

Rudin 9.7

Suppose first that f is defined on $E \in \mathbb{R}^n$ and $D_i f$ are bounded in E, $1 \le i \le n$. We have that $D_i f = \frac{\partial f}{\partial x_i}$.

f is cont if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t $\|f(x) - f(p)\| < \varepsilon$ for all $\|x - p\| < \delta$

We know that each $D_i f$ is bounded on E.

$M = \sup\{\|D_i f\| : x \in E\} < \infty$

We can use MVT to show that for any point $x, y \in E$,

$|f(x) - f(y)| \le M|x-y|$

By letting $\varepsilon = M\delta$. Now $\forall \delta > 0$ whenever $|x-y| < \delta$ we have

$|f(x) - f(y)| < \varepsilon$

3. Let $E \subset \mathbb{R}^2$ be any closed subset. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be def by

$$f(x) = \begin{cases} 0 & x \in E \\ \inf\{|x-y| : y \in E\} & x \notin E \end{cases}$$

$\inf\{|x-y|\}$ exists and is cont since E is closed.

Then by construction, $f^{-1}(0) = E$ and since $f(x)=0$ $x \in E$ is cont it is cont.

4. The idea is that from $f(x,y)$ we can isolate y w.r.t. x as $g(x,y)$ where $f(x,y)=0$, This is useful for the point where $f(x,y)=(0,0)$